

## Path Curves

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### Introduction

Geometry means measuring the Earth. It was certainly known to the Ancient Egyptians for they used it to re-establish land boundaries after the Nile had flooded, and possibly before. However the systematic development of the subject comes to us from the Ancient Greeks, particularly Euclid. He set forth a number of axioms that are supposed to be self-evident and therefore in no need of proof e.g. given two points then there is one straight line joining them. These were used to prove other theorems such as that of Pythagoras (which was already known to the Egyptians and used practically). Most of his geometry is based on the notions "length" and "angle" and much effort goes into proving that distinct lengths are or are not equal, and likewise for angles e.g. an isosceles triangle has two equal sides, and it is then proved that it also has two equal angles.

In what follows we will illustrate our subject rather than prove all the results formally. Further details are available in the references. The methods of George Adams and Lawrence Edwards will be used extensively.

### Projective Geometry

The Greek mathematician Pappus proved the following theorem:

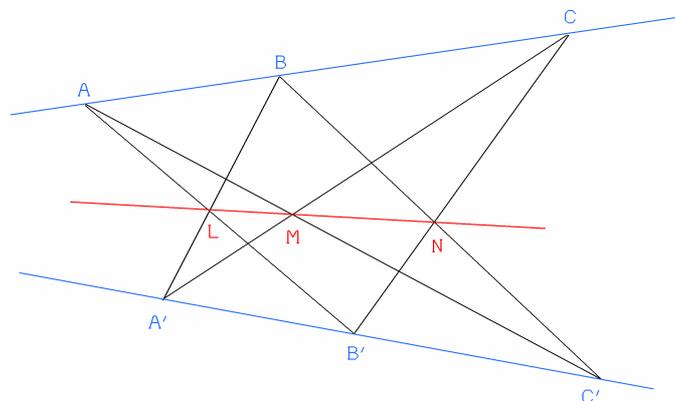


Figure 1

We take any two (blue) lines and any three points on each ( $A B C$  and  $A' B' C'$ ). We join  $AB'$  and  $A'B$  meeting in the point  $L$  as shown. Similarly  $AC'$  and  $A'C$  meet in  $M$ , and  $BC'$  and  $B'C$  in  $N$ . Pappus proved that the three points  $L M$  and  $N$  always lie on a straight line no matter how we choose the initial lines and points. Unlike Euclid's approach, this theorem makes no use of lengths of lines or angles between them: it only concerns relationships. It was the first theorem of what centuries later was developed further as projective geometry, which in the same spirit makes no use of lengths and angles (at least in its pure form).

The architect Girard Desargues in the 17th Century proved the next very fundamental theorem:

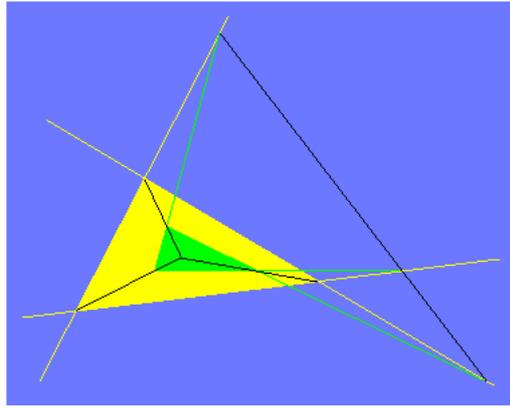


Figure 2

Two triangles are shown, one yellow and one green. They are such that when corresponding vertices are joined, the three resulting black lines meet in a point. Then when corresponding sides are extended to meet, the three resulting points lie on a straight line. The converse is also true: if corresponding sides meet in this way then the joins of the vertices are concurrent. This is what is known as a self-dual theorem, for if three lines meet in a point then three points lie on a line, and *vice versa*. In other words point and line are dual: a true statement about points and lines implies a dual true statement where their roles are reversed.

Projective geometry developed rapidly in the following centuries, and many beautiful theorems were found all based on relationships and with little regard for lengths or angles. The two most fundamental operations performed are *projection* and *section*:

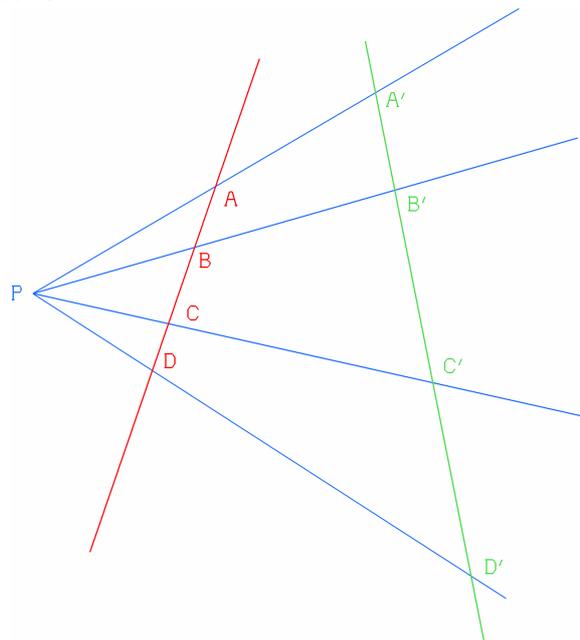


Figure 3

*Projection* is the operation of projecting the points A B C D on the red line from the point P, the centre of projection. The result is a set of blue lines called a *flat pencil*. Any number or all points of the line may be projected. *Section* is the dual operation of cutting a pencil with a line, see the green line above. A set of points on a line is dual to a pencil of lines in a point, and is called a *range* of points, so we have projected a range A B C D into a pencil and then taken a section to produce a second range A' B' C' D'. Two such ranges are said to be *in perspective*. Much of projective geometry is based on these two operations. If we

go further and project the second range from some other point Q to produce a second pencil then the latter is in perspective with the blue pencil: perspectivity is a dual concept.

If we repeat these operations a number of times we end up with a final range A'' B'' C'' D'' which is not now in perspective with A B C D, but is said to be *projective* to it. In other words a number of combined perspectivities yield a projectivity. If the pairs of points A A'', B B'', C C'' and D D'' are joined the resulting lines envelope a *conic section* such as an ellipse or a hyperbola. This is illustrated below where we have three ranges (and two pencils) and corresponding points of the first and last are joined by magenta lines which are tangential to an ellipse:

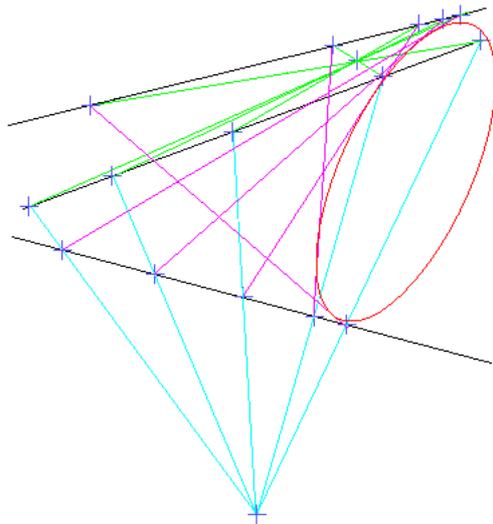


Figure 4

The process of finding the points on the final line corresponding to those on the first is also called a *projective transformation*, and the first range (and its line) is said to be transformed into the final one. The dual transformation would transform one pencil into another, and corresponding lines would meet in points lying on an ellipse or other conic section.

Measures

The following diagram illustrates how this may be applied to transform a line into itself:

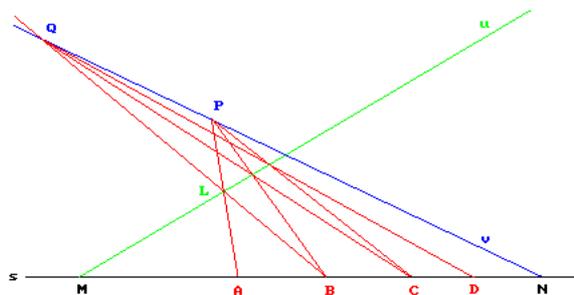


Figure 5

We start with the black horizontal line s and choose a point A lying on it. We project A from a centre P, and then take a section with a fixed green line u to give the point L. Then we choose a second centre Q and

project L back onto  $s$  again, giving the point B. We repeat the process, projecting B from P and then projecting from Q the resulting point on  $u$  back onto  $s$  again to give the point C. And so on ... The blue line QP meets  $s$  in the point N. Let us see where N goes: we project it from P to give the point where the blue line intersects  $u$ , and then project that from Q back onto  $s$  again, only to end up where we started! Thus N is a *self-corresponding point*. It can be seen that the range A B C D is starting to get more crowded towards N, and if we proceed with further points E F ... after D we find they get closer and closer together without ever reaching N. They cannot reach N for the following reason. The process is reversible, so if we start with D, then we may project it from Q to give a point on  $u$ , and then project that onto  $s$  from P, taking us backwards to the point C. So, if some point Z long after D could reach N on the next step, then we could reverse the process and go backwards from N to Z, which is not possible as N stays put.

Similarly the point M where  $u$  meets  $s$  is self-corresponding as is easily seen. So transforming  $s$  into itself gives a remarkable result: two of its points are fixed points and all others move along the line as the transformation is repeated, with a "gesture" of expansion and contraction. For a point starting off near M moves to the right in small steps that increase until it is near B and then the steps get smaller again as it approaches N. If we take points on the right beyond N the steps increase to enormous size and then the point reappears at the other end of  $s$  and heads in towards M ! This should be tried out; the transform is easy to construct. The idea of transforming something into itself is very fundamental in projective geometry and of great importance as we shall see. The dual transformation is that of a pencil into itself, and indeed the red pencils in P and Q above are undergoing just that. The dynamic of this transformation is called a *measure*, and since it has two fixed points and "breathes between the two" it is sometimes referred to as a *breathing measure*, or more formally a *hyperbolic measure*. Thus the term "measure" is used to refer to the transformation process and its terms (points or lines). Later we will see that there are also measures of planes in three dimensions.

Referring back to Figure 5, we could arrange for the lines  $u$  and  $v$  to meet on  $s$ , in which case the points M and N would coincide. The resulting measure is called a *step measure*, or academically a *parabolic measure*. The reason for this name is that if we move the point  $M=N$  indefinitely far to the left eventually the two lines  $u$  and  $v$  will become parallel to  $s$ . It is as if the three lines still met even though they are parallel, and we say they meet in an *ideal point*. It is thought of as a point because the configuration behaves just as if the three lines really met. Now in this case the steps of the measure all become equal, which gives rise to its name:

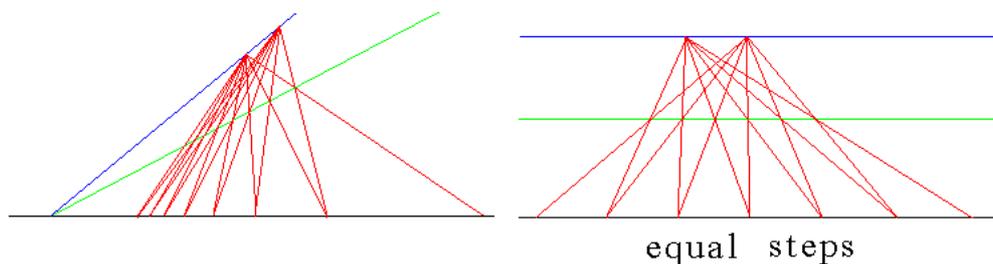
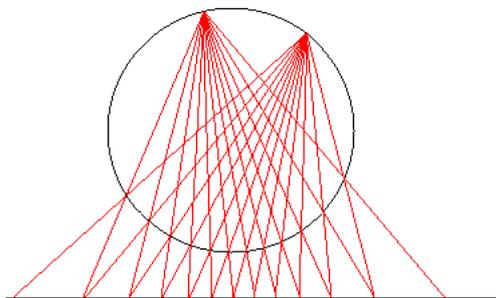


Figure 6

The situation with  $u$  and  $v$  meeting on  $s$  is shown on the left, together with the step measure, and then the parallel case is shown on the right. In passing, we could have moved  $M=N$  to the right until  $u$  and  $v$  became parallel, and we would end up with the same result and therefore the ideal point would be the same. Accordingly it is said in projective geometry that there is only one ideal point in which parallel lines meet (not two, one at either end!). Such points are also thought of as being at infinity. But using the concept "at" is problematic without considerable explanation and we will stay with the concept of ideal points.

There is a third possibility: that the measure has no fixed points. This is called a *circling measure*, or more formally an *elliptic measure*. It is easiest to construct using a circle instead of the two lines  $u$  and  $v$ :



no fixed points

Figure 7

The two pencils of red lines are also in circling measure.

It might be thought there could be three or more fixed points, but it turns out that if a third point is self-corresponding *then all points on the line are self-corresponding* i.e. the whole thing freezes. The result is called the *identity transformation*. This is easily seen if we reverse the process in Figure 5 and suppose we start with the fixed points  $M$  and  $N$  on  $s$ , and then take  $A$  and  $B$  as corresponding points. We then choose any line  $u$  containing  $M$ , and  $v$  containing  $N$ , and join  $A$  to some point  $P$  on  $v$  meeting  $u$  in  $L$ . Finally we join  $B$  and  $L$  meeting  $v$  in  $Q$ , and we find we have set up a transformation that leaves  $M$  and  $N$  fixed while transforming  $A$  into  $B$ . Now suppose that  $A$  is also fixed. To set this up we proceed as above, but since  $B=A$ , the join of  $B=A$  and  $L$  meets  $v$  not in a separate point  $Q$  but in  $P$  again. We then find that with only one centre of projection every point on  $s$  is transformed into itself, and we see why there can be at most two fixed points for a proper measure.

Transforms of the Plane

If we can transform a line into itself, it is natural to extend the idea to transforming a whole plane into itself. This is illustrated below.

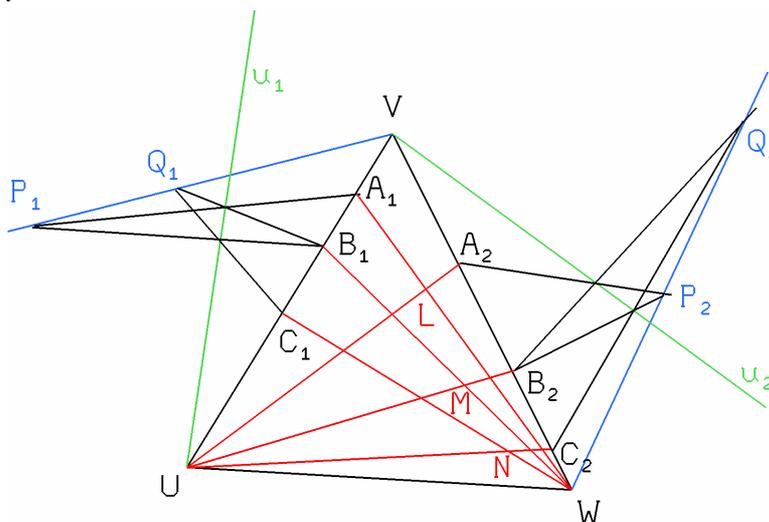


Figure 8

This looks complicated but we will soon find our way through it. We have a triangle UVW and on the side UV we have set up a transformation as in Figure 5, with points  $A_1 B_1 C_1$  being transformed using the line  $u_1$  and the centres of projection  $P_1$  and  $Q_1$ . The fixed points are now U and V. Similarly for the side VW we have centres  $P_2$  and  $Q_2$  together with the line  $u_2$  transforming  $A_2$  to  $B_2$  to  $C_2$ . W is a third fixed point, and generally a plane has three fixed points. The line  $A_1W$  transforms into  $B_1W$  as  $A_1$  moves to  $B_1$  as W is fixed. Then  $B_1W$  moves to  $C_1W$ . When  $A_1$  moves to  $B_1$ ,  $A_2$  moves to  $B_2$  and thus the line  $UA_2$  moves to  $UB_2$ , as U is a fixed point. The point L where  $UA_2$  and  $WA_1$  meet thus moves to M where  $UB_2$  and  $WB_1$  meet. Finally M moves to N where  $UC_2$  and  $C_1W$  meet. This shows how points in the plane are transformed, and some special lines. Had we chosen L randomly then we would join W to L in a line meeting the line UV in the point  $A_1$ , and also we would join U to L meeting VW in  $A_2$ . Then we would transform  $A_1$  to  $B_1$  and  $A_2$  to  $B_2$  and finally the lines  $UB_2$  and  $WB_1$  would meet in M, the transform of L. Thus every point in the plane can be transformed in this way. Suppose we wanted to transform a line. It would meet UV and VW in points  $A_1$  and  $A_2$  respectively, say (line not shown). We then transform  $A_1$  and  $A_2$  giving the transformed line as  $B_1B_2$ . Thus every line in the plane can be transformed. We note that the lines UV and VW are fixed lines that transform into themselves since U V and W are fixed. Clearly UW is also fixed, so duality holds true, there being three fixed lines as well as three fixed points. Thus transforming a plane into itself leaves a triangle fixed, although the points other than U V and W move along its sides. This is the general case, where we have a triangle with breathing measures on its sides. We will see that there are other possibilities with other measures.

Path Curves

But first of all we now introduce the idea of *path curves*. Returning to Figure 8 the successive points L M and N are not quite on a straight line; in fact they are on a curve starting at V, passing through L M N and ending up at W. Such a curve along which successive points of a transform move is called a *path curve*. The following diagram shows the dual approach to path curves which is initially simpler:

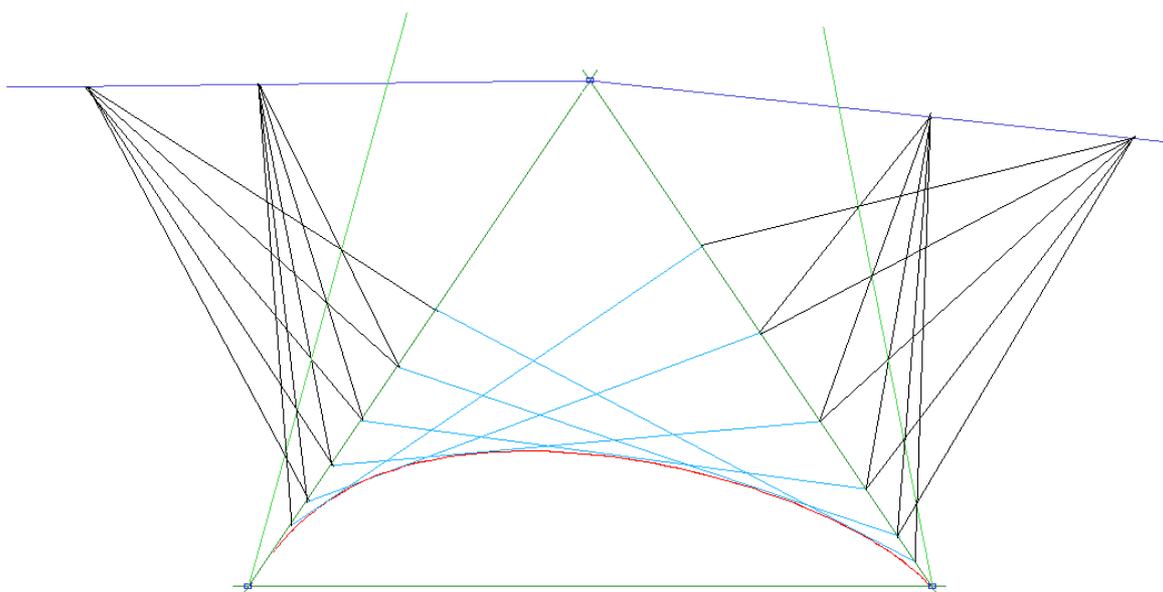


Figure 9

We show the triangle again with the two breathing measures on two of its sides. However we join corresponding points of the measures, noting that as one moves up the other moves down. The blue joining lines touch (or envelope) a red curve which is the path curve. We have used fairly large steps but we could

in principle choose much smaller ones until eventually the whole curve would be enveloped. Any tangent to the curve is transformed into another tangent to it. Dually any point on it is, using the method of Figure 8, transformed into another point also lying on it. Thus although all except the fixed points are moving in the plane, the path curve itself is a fixed curve. We could have joined the lowest point on the left to the second highest on the right instead of the highest, and proceeded from there, which would have given another path curve of the family. Indeed the triangle is filled with path curves none of which cross each other. For example choose a point of intersection of two blue lines above the path curve and then follow the diagonals of the lozenges and another path curve is evident to the imagination. The whole field of path curves inside the triangle is indicated as follows:

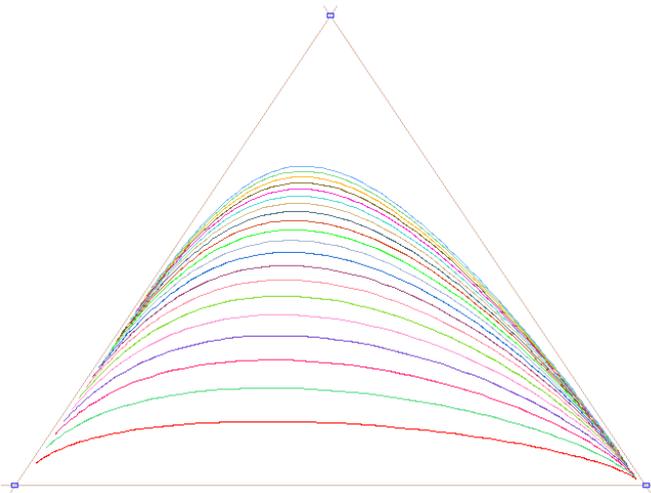


Figure 10

Egg Forms

In Figure 9 we can move the top vertex of the triangle upwards until eventually the two lines UV and WV become parallel i.e. V has become an ideal point, and then turn the diagram through 90°. The result looks like this:

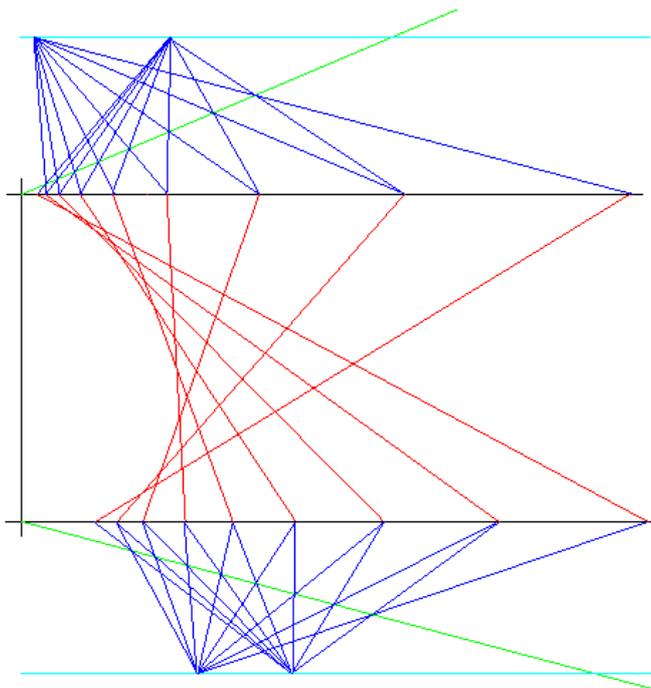


Figure 11

Note that the lines on which the centres of the blue pencils lie are now parallel to the top and bottom invariant lines. One fixed point of each of the breathing measures is now ideal, illustrating again how such points act like ordinary ones even if they are inaccessible. As a result the points of each measure are such that their distances from the vertical axis are in geometric progression (explained fully later).

The path curve is now clearly egg-shaped. In fact it accurately describes many forms found in Nature such as birds eggs, plant buds, pine cones and the human heart. We will return to the practical research after we have shown the three dimensional aspect which gives complete eggs.

Vortices

In Figure 11 the measures on the top and bottom fixed lines move in opposite directions so that the innermost point of one is joined to the outermost of the other. Now we replace the lower one with a breathing measure on the vertical axis:

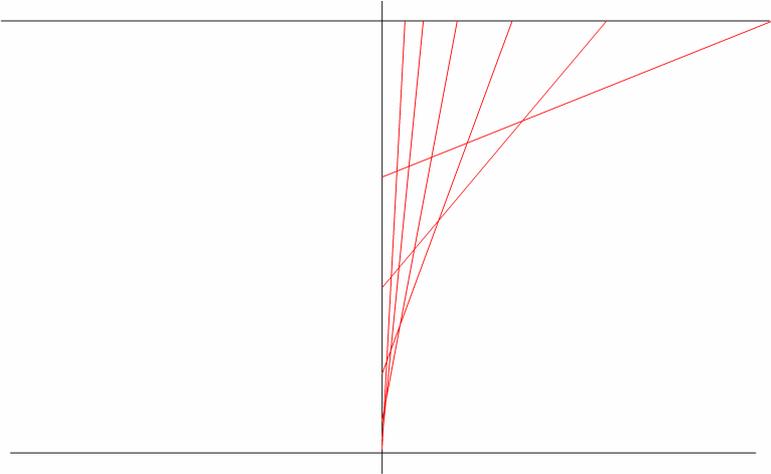


Figure 12

We do not show the construction of the measures now that we are used to the process. The result is a vortical path curve which tends towards the lower fixed point on the vertical axis, and towards the ideal point on the top fixed line. We may now vary this in two ways: by moving the top invariant point on the vertical axis upwards until it becomes an ideal point, or instead moving the lower one downwards until it becomes ideal. The results are shown below:

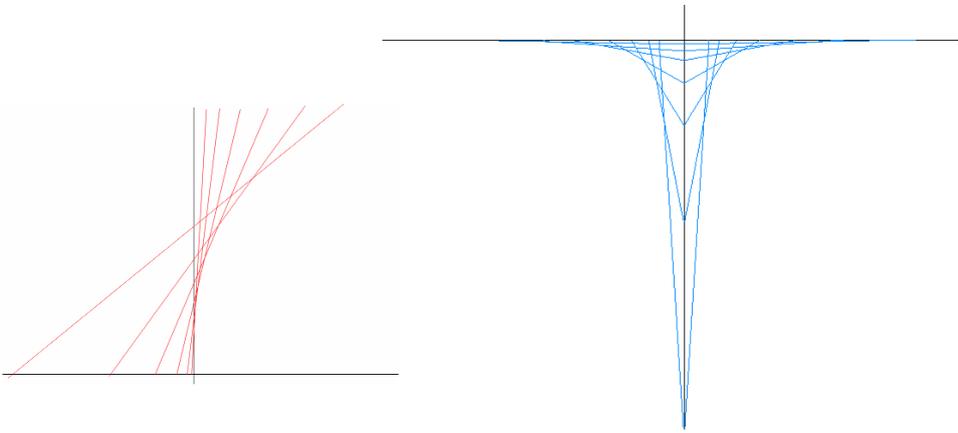
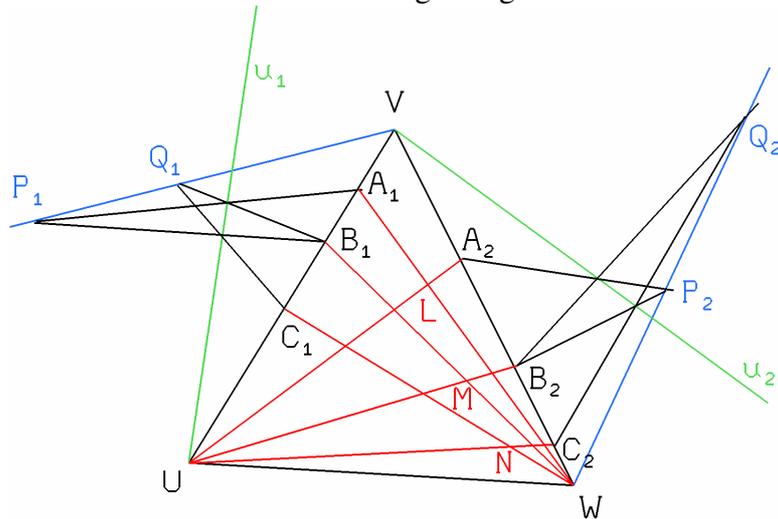


Figure 13

The path curve on the left starts from the lower fixed point and expands outwards and upwards towards infinity, while the pair on the right pass from the ideal point on the horizontal fixed line to that on the vertical one. This path curve when suitably proportioned accurately fits actual water vortex profiles.

Spirals

So far we have only used breathing measures. However when we pass to three dimensions we will need the case where a circling measure is used instead. Referring to Figure 8 which we show again for convenience:



note that the breathing measure on UV is projected from W onto the lines UA<sub>2</sub> UB<sub>2</sub> and UC<sub>2</sub> i.e. all the lines of the flat pencil in U have a similar breathing measure between U and the point where they meet VW. We now replace the breathing measure of lines centred on U with a circling measure of lines. There are now no fixed lines so UV and UW no longer exist, and thus the line VW remains but without V and W. However the breathing measures on the lines through U still exist. The result is shown below:

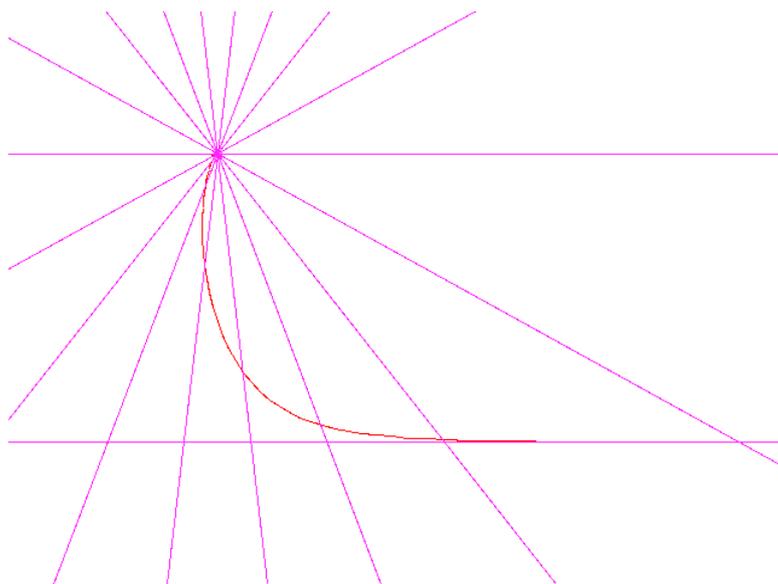


Figure 14

The circling measure of lines is shown together with one path curve which starts from the centre of the pencil and tends indefinitely towards the fixed line. It crosses the lines of the pencil in successive points of the breathing measure on each. If we now move the fixed line downwards until it becomes ideal (an ideal line is one all of whose points are ideal) then we get logarithmic spirals:

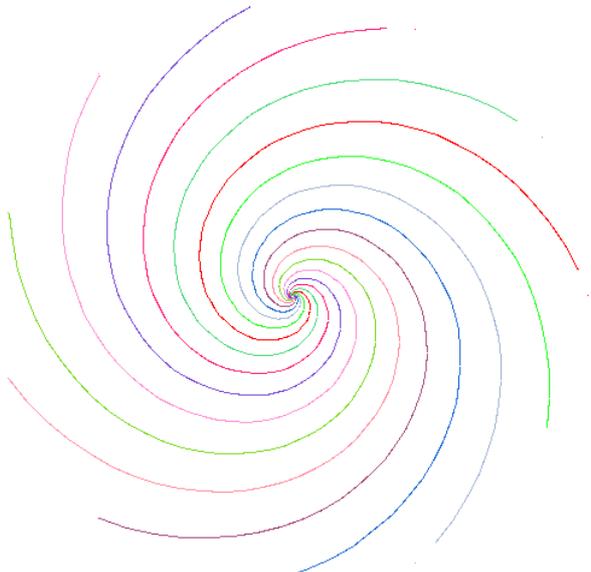


Figure 15

Geometric Progression and Spirals

In connection with Figure 11 we mentioned that the breathing measures were special because one of their fixed points is ideal.

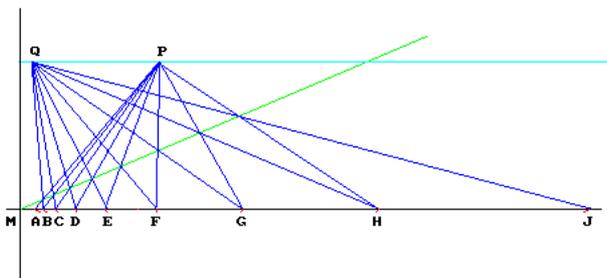


Figure 16

What this means is that the ratios  $MA/MB$ ,  $MB/MC$ ,  $MC/MD$  ... are all equal, so there is a definite number or multiplier which applied to the length  $MA$  will give  $MB$ , then to  $MB$  gives  $MC$  and so on. Such a sequence is known as a *geometric progression*. This departs from our initial claim that projective geometry does not concern lengths. As soon as ideal points are included a connection with so-called *affine geometry* is made. Affine geometry transforms ideal points only into other ideal points; it never transforms them into ordinary points. Conversely it never transforms ordinary points into ideal ones. The above construction is affine because the ideal point on the parallel lines is fixed i.e. it is transformed into itself and so remains ideal. However only *ratios of lengths* are significant in affine geometry, not lengths in isolation. This is because an affine transformation preserves ratios but not lengths, as is clear in the above transformation where  $MA$  is transformed into  $MB$  which has a different length from  $MA$ , so length is not conserved. This special form of the breathing measure is also known as a *growth measure*.

The logarithmic spirals in Figure 15 cut a line through their centre in a geometric progression as we illustrate below:

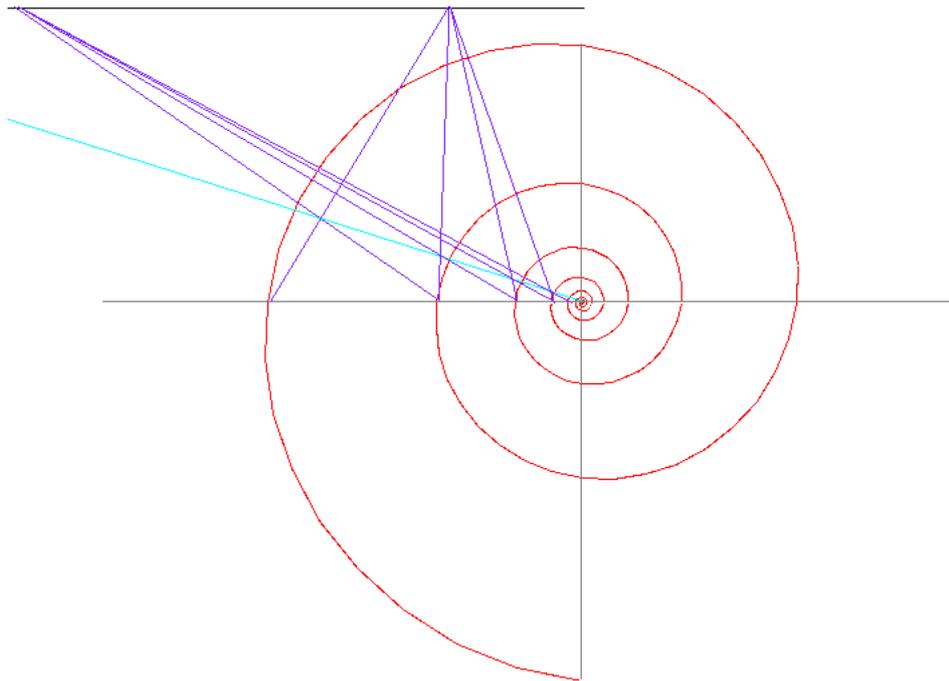


Figure 17

This spiral winds out more slowly than those in Figure 15 to enable our demonstration to work, but it is of the same family. A geometric breathing measure has been constructed which shows that the spiral cuts a line through its centre in a geometric series. This is indeed true of any line at any angle through the centre. This spiral is ubiquitous in Nature being seen in Nautilus Pompilium shells, sunflowers and so on, and is another case of a path curve that describes natural forms. It is not merely a "form fitting" exercise as we will show later, but a demonstration and a test of some important scientific ideas.

Polar Transformations

So far we have studied transformations which move points to other locations and lines to other lines. It is also possible to relate points to lines and *vice versa*, giving *polar transformations*. The simplest case was already implicit in Figure 3: by projecting a range we transformed points into a pencil of lines. However this is a rather limited example and a better one is shown below:

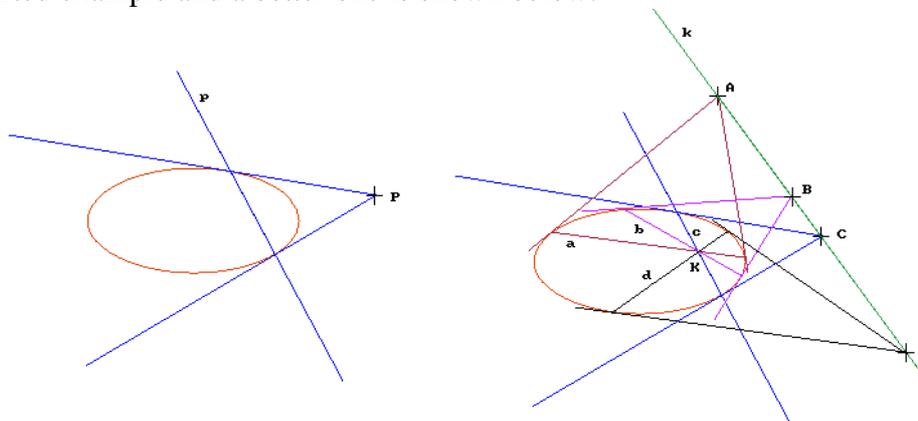


Figure 18

On the left we transform the point P into the line p using an ellipse. The method is to draw the two tangents to the ellipse through P, and then join the two points of tangency. P is called the *pole* of p, and p the *polar* of P. Any point outside the ellipse may be transformed into a line in this manner. Conversely if we wish to find the pole of a line we draw the two tangents where it cuts the ellipse to give the pole as their point of intersection. Thus the transformation is reciprocal.

On the right we show a line k containing four points A B C and D. The polars a b c and d respectively are seen to meet in a point K so that a flat pencil is polar to a range and *vice versa*. K and k are pole and polar. This solves the problem of finding the polar of a point K lying inside the ellipse, for we take any two lines through K, find their poles and join the resulting two points to give k. Evidently we have also solved the problem of finding the pole of a line that does not cut the ellipse. Now we can say that every point in the plane has a polar line and *vice versa*, so the whole plane has been transformed. A point P lying on the ellipse is polar to the tangent at P and *vice versa*. Such pairs are special in that they are *incident* i.e. the point lies on its polar line. In fact any conic section may be used to establish a polarity i.e. a circle, ellipse, hyperbola or parabola.

A transformation relating points to points and lines to lines is formally called a *collineation*, while a polarity is an example of a *correlation*.

We may use two polarities alternately so that a point P is transformed into a line p by the first, and then p is transformed into a new point Q by the second, and so on. The points P Q ... lie on a path curve, and the lines p ... envelope a distinct path curve as illustrated below:

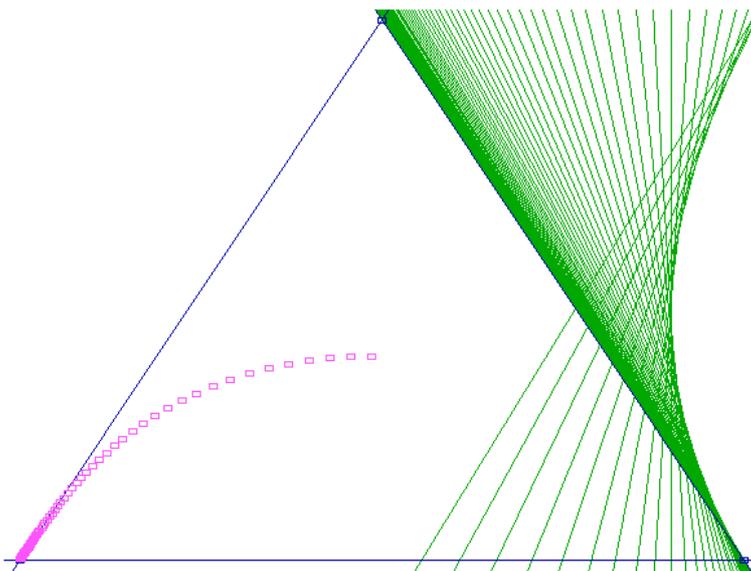


Figure 19

The polars of complete figures can be found using the constructions of Figure 18, and for example conic sections such as ellipses transform into conic sections.

## Three Dimensional Projective Geometry

We worked with ranges and flat pencils in the plane, and found measures in both. In space a number of planes may share a common line, giving an *axial pencil* of planes:

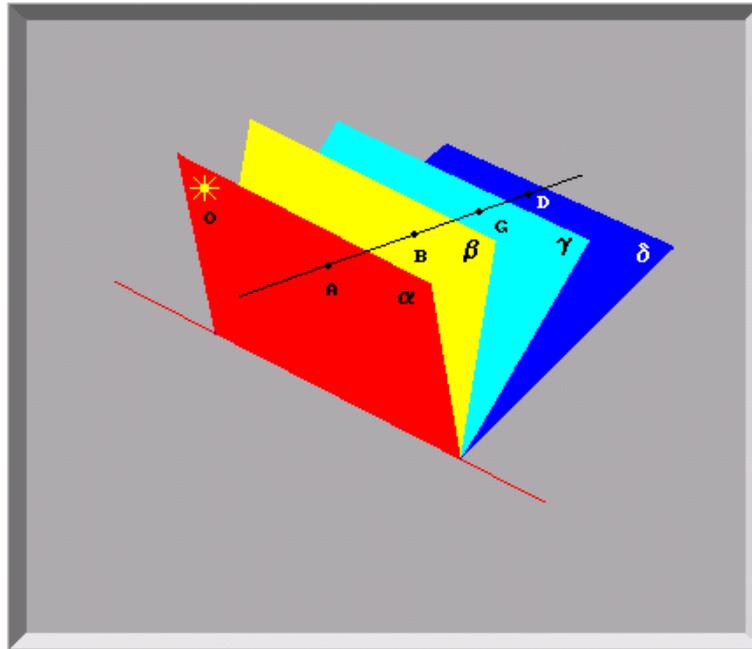


Figure 20

The common line of the pencil is called its *axis*. As shown another line intersects an axial pencil in a range of points, and dually a plane would intersect it in a flat pencil of lines. We may relate two axial pencils in two ways: by making them share a range of points or a flat pencil of lines. If they share a flat pencil then their axes must intersect, and if in that case the plane containing their axes is self-corresponding then the pencils are said to be in perspective. If they share a common range and their axes intersect in a self corresponding plane, then again they are in perspective. Otherwise they are projective. If their axes are *skew* (i.e. do not intersect) then they cannot have a common plane and so if related they are projective. Two projective axial pencils intersect in a *regulus* i.e. corresponding planes intersect in lines lying on a regulus illustrated below:

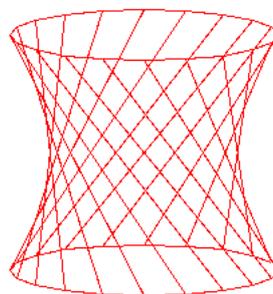


Figure 21

A regulus is a system of lines which in fact extend to infinity, lying in a surface called a *ruled hyperboloid*; "ruled" because its surface contains straight lines. If the axes of two axial pencils intersect but they are not in perspective then this surface becomes a cone.

In three dimensions we may study transformations of the whole of space into itself. One measure defined a one dimensional transformation and two a two dimensional one, so we expect a three dimensional one to require three measures, and that is in fact true. Before proceeding we should note that projective transformations belong to a special variety called *linear transformations*. This is because straight lines always transform into other straight lines, never into curves, and planes always transform into planes, never into curved surfaces. Also *incidence* is preserved i.e. if a point lies on a line then the transformed point will lie on the transformed line, and similarly for points and lines in planes. There are other interesting transformations where this is not true but they do not concern us just now.

In the plane we had an invariant triangle, and in space there is an *invariant tetrahedron* with three defined breathing measures on three of its edges:

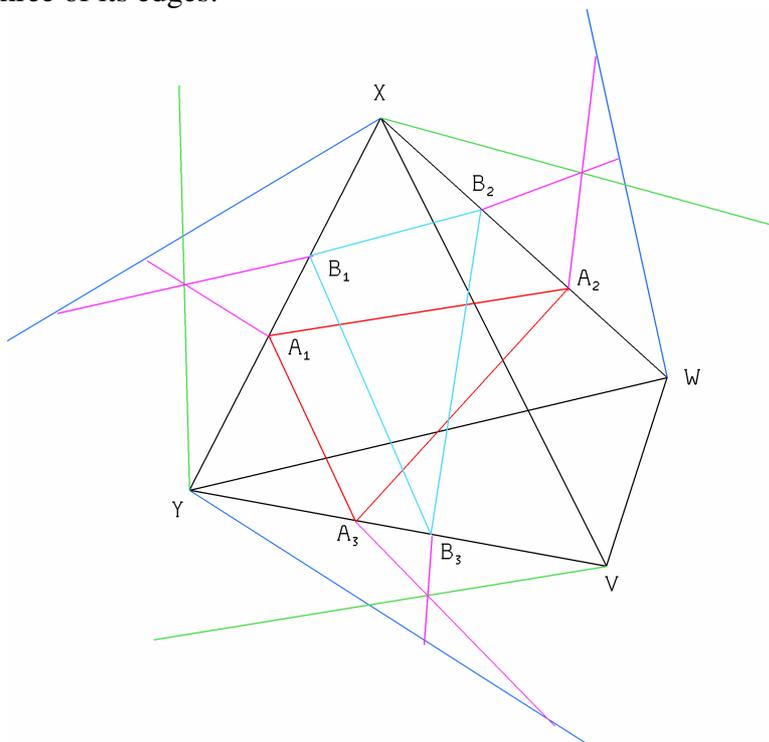


Figure 22

The vertices VWXY of the tetrahedron are fixed points, and generally there are four except in special cases. We show three breathing measures on the edges XY, YV and WX. To transform a plane such as the red one above we find the three points  $A_1$ ,  $A_2$  and  $A_3$  where it meets the lines XY, WX and YV. Then the points  $A_1$ ,  $A_2$  and  $A_3$  are transformed by the measures into  $B_1$ ,  $B_2$  and  $B_3$  which defines the cyan plane above into which the red plane is transformed.

The easiest way to transform a point is to take three planes containing it and find their transforms. The transformed point will be the common point of the latter. Thus we take the plane containing VW and the point P, and VY and P, and XW and P. The plane VWP intersects XY in a point  $A_1$  which is transformed into  $B_1$  by the measure, giving a new plane  $VWB_1$ . Similarly  $VYP$  transforms into  $VYB_2$  and  $XWP$  into  $XWB_3$ . Then the point Q into which P is transformed is the point common to the planes  $VWB_1$ ,  $VYB_2$  and  $XWB_3$ . This is not so easy to illustrate clearly. Another approach is shown below.

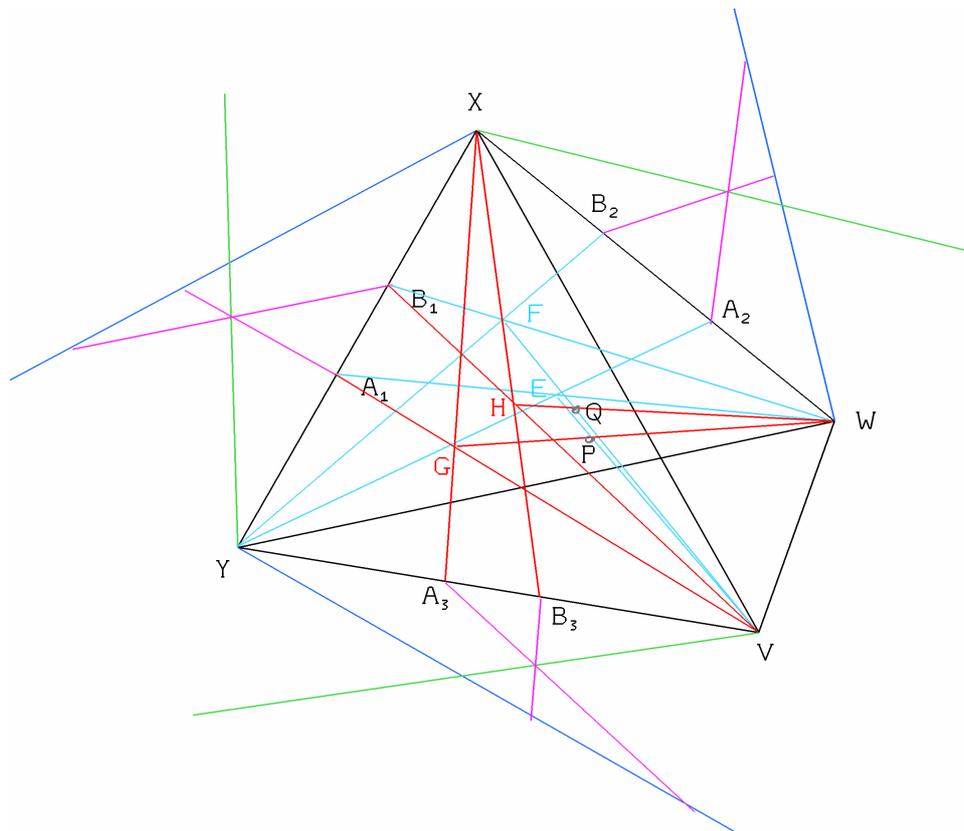


Figure 23

Concentrate on the red lines first. We draw the line  $WP$  meeting the plane  $XYV$  in  $G$ . Now join  $VG$  meeting  $XY$  in  $A_1$  and then find  $B_1$  using the measure. Then join  $XG$  meeting  $YV$  in  $A_3$ , find  $B_3$ , and join  $XB_3$ . Since the lines  $VA_1$  and  $XA_3$  transform into  $VB_1$  and  $XB_3$ , the latter two lines meet in a point  $H$  which is the transform of  $G$ . Finally the line  $WH$  is the transform of  $WG$ , so  $Q$  must lie on  $WH$ . A similar procedure is followed for the cyan lines transforming the line  $VE$  through  $P$  to  $VF$ , so that  $Q$  must lie on  $VF$ . Thus  $Q$ , the transform of  $P$ , is the point of intersection of  $WH$  and  $VF$ . Every point in space may be transformed into another by this means or the previous approach with planes, and we have seen that every plane may be transformed. That leaves lines. They are simple now. Suppose the line we want to transform meets the plane  $XYV$  in the point  $G$  in Figure 23, and  $XYW$  in  $E$ . Then we transform  $G$  to  $H$  and  $E$  to  $F$  as before, giving the join of  $H$  and  $F$  as the transformed line.

If we then transform  $Q$  into a series of subsequent points we will end up with a three dimensional path curve. It will be a *twisted curve* in contrast to those in Figure 10 as it does not lie in a plane. We will illustrate some significant such curves in a moment when we have made connection with our previous study of egg profiles. We need a special form of tetrahedron for that, and as eggs are circularly symmetrical about an axis we expect circling measures to be involved. Figure 11 gives us a clue as to what we need. We see a vertical invariant line which will be the axis of symmetry of our complete three dimensional egg, and the two parallel horizontal invariant lines now represent two invariant planes seen edge-on. The growth measures in the original lines are now those arising when they cut logarithmic spirals in the parallel invariant planes (as in Figure 16). This is illustrated in the following diagram:

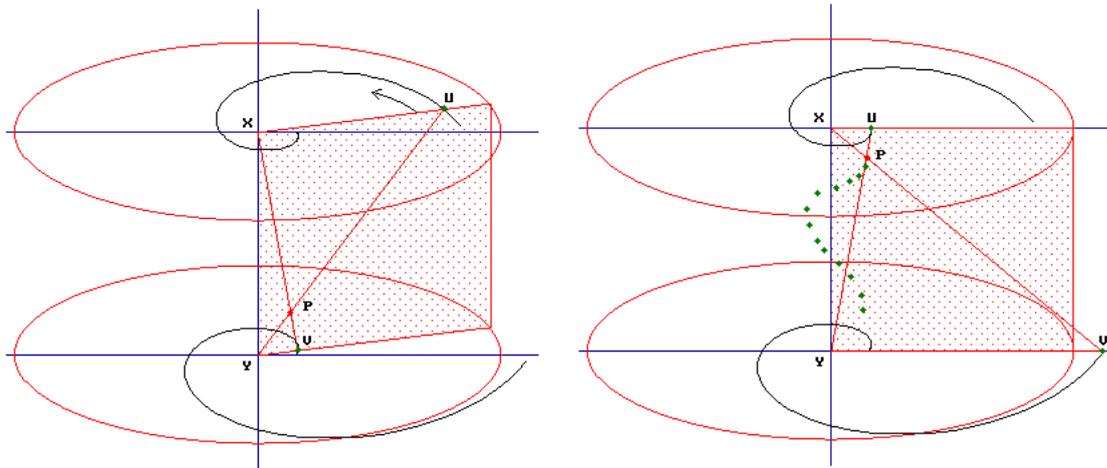


Figure 24

On the left the original triangle is shown in blue, with the vertical invariant line and the two invariant points X and Y on it. Referring to Figure 22 (or 23) we have moved XY into the vertical and the planes XVW and YVW have become parallel and at right angles to XY, with VW as an ideal line. However we have taken two circling measures one in each plane indicated by the fact we draw the planes as circles (in perspective). The lines XV XW YV and YW are missing, as happened to the lines UV and UW when we passed from Figure 8 to 14. The tetrahedron is hardly recognisable now! The path curves in our top and bottom planes are thus spirals as shown, noting that they wind in opposite senses. Now we come to the way the transformation works. If we wish to transform a point P we take the plane XYP (shaded) and join XP meeting the lower plane in V and YP meeting the upper plane in U. There will be one spiral path curve through U in the upper plane and one through V in the lower one (c.f. Figure 15: there is just one path curve through every point of the plane). We have selected the circling measures so that the spirals wind outwards in opposite senses. Recalling that the lines of the circling measure rotated about the invariant point in Figure 14, we expect the lines XU and XV to rotate about X and Y respectively i.e. the shaded plane rotates about XY as we repeat the transformation, U then following its spiral inwards toward X and V moving outwards along its spiral away from Y. The plane rotates in the direction of the arrow as we keep repeating the transformation so that P moves upwards since U moves in towards X and V outwards away from Y. A series of positions of P are shown in the diagram on the right where the plane has completed one rotation. They lie on a spiral path curve winding about the axis XY. Were we to continue, P would wind round and round XY moving ever closer to X. On the other hand had we gone backwards P would have moved ever closer to Y. It is clear to see in this example that the shaded plane can rotate continuously and thus produce a continuous path curve.

Had we held the shaded plane fixed and rotated the tetrahedron about XY then P would have moved in that plane to form an egg-shaped profile as in Figure 11. Recalling from Figure 17 that a spiral path curve relates to a growth measure, the successive positions of U in the shaded plane form an inward-moving growth measure while those of V form an outward moving one, so effectively we have the construction of Figure 11 taking place in the shaded plane. Thus we see that the spiral is an egg-spiral. We show below an egg with many such path curves, recalling that the circling measures create many spirals as in Figure 15, which then give us many egg spirals.

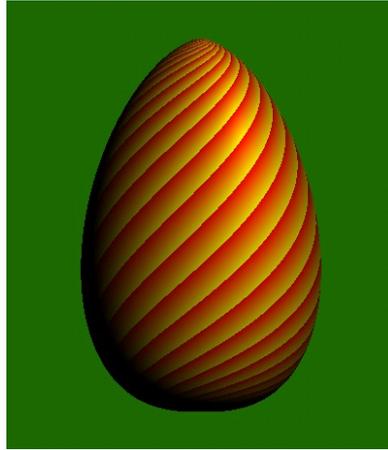


Figure 25

The next figure shows the path curves of such a system related to a pine cone, as produced by Lawrence Edwards (Reference 3):

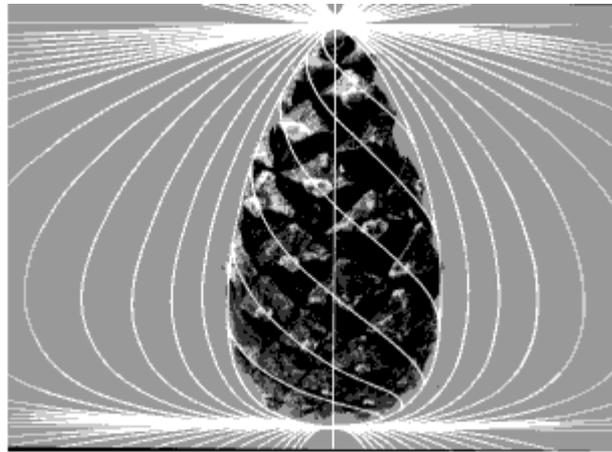


Figure 26

The profile is clearly a path curve belonging to a whole "field" of such curves, and the spiral versions follow the spirals of the pine cone.

Had the spirals wound in the same sense we would have obtained a vortex spiral as this would be equivalent to Figure 12 in the shaded plane:

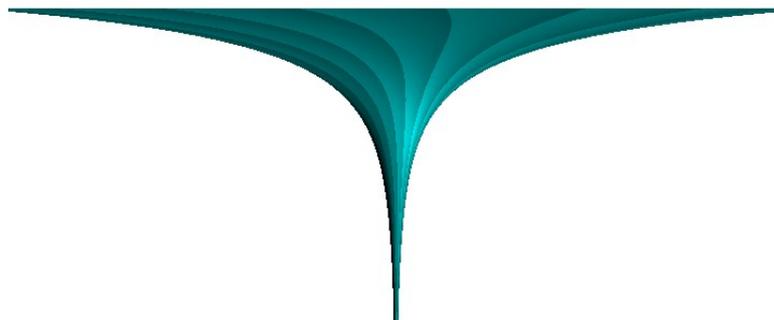


Figure 27

## Dual Approach

It will be important later to understand the dual construction of path curves in three dimensions. Referring to Figure 24, we would start with a plane rather than a point P and transform that:

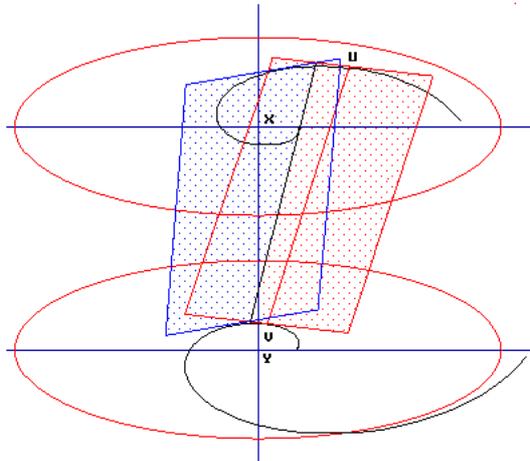


Figure 28

The initial red plane is shown shaded, intersecting the upper fixed plane in a line, and there is one spiral tangential to that at U. Similarly in the bottom plane there is one spiral tangential at V. The red line joining U and V is called the *path line*. The transformations in the top and bottom planes will move U and V to new positions at which two new parallel (blue) tangents will exist defining the new blue plane corresponding to the red one. After many repetitions of the transformation as before, a set of planes arises defining a path curve of the same family i.e. it looks just like the one in Figure 24. This is more difficult to visualise as each plane is tangential to that curve in a special way such that it is an *osculating plane*.

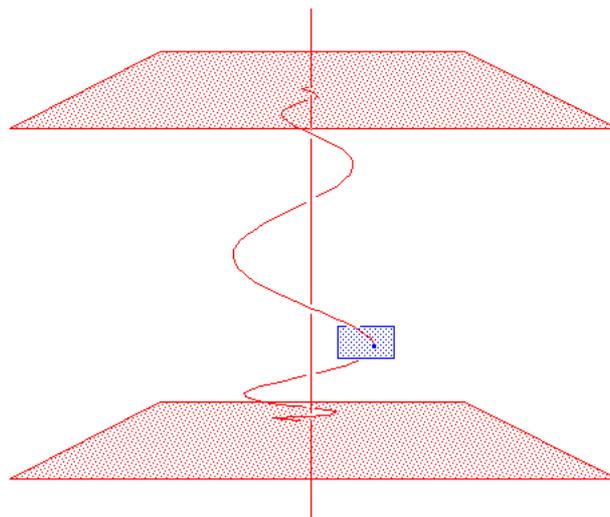


Figure 29

Here we have shown a path curve with a blue osculating plane where the curve touches that plane at the blue point, passing behind the plane as it goes downwards. Strictly speaking the planes form what is called a *developable* which is the dual of a curve of points. Such a developable has a well defined edge known as the *cuspidal edge* which in this case is the path curve. What will be important later is that if we take a

definite point on the curve and an osculating plane tangential to the curve at that point, like the blue pair above, then the transformation of space will move the point and plane simultaneously along the same path curve. The curve is woven dually of planes and points. It is important to emphasise that although the pointwise approach is much easier to visualise, it is only half the story.

Parameters

When cooking a meal in an oven there are two important quantities: what the temperature is and how long the meal is left in the oven. The temperature is controlled by e.g. the gas mark which typically varies from 0 to 9. This number or quantity is called a *parameter*, the other being the time in minutes. Parameters are overall settings that control a process, usually expressed as suitable numbers.

It will be noticed that the egg forms have a greatest radius and are blunter at one end and sharper at the other. Referring back to Figure 11 we chose two different growth measures on the top and bottom fixed lines which is why this happens. If the growth measures are the same then the path curve profile becomes an ellipse. However as the growth measures become more and more different the egg becomes ever more pointed and the height of the maximum radius moves down from the central position between the fixed planes. The type of egg curve is determined by a parameter symbolised by the Greek letter  $\lambda$  which is easily calculated as follows:

$$\lambda = \frac{d-h}{h}$$

where  $d$  is the distance between the invariant lines (or planes in the three dimensional case) and  $h$  is the height at which the maximum radius occurs. Thus if  $h=d/2$  giving an ellipse then  $\lambda = 1$ , and as  $h$  becomes ever smaller  $\lambda$  becomes ever larger. A similar situation occurs for vortices, but for them  $\lambda$  is a negative number which is not of course related to a maximum radius. There is a deeper definition of  $\lambda$  which we will come to later which embraces both types of path curve. The following diagram illustrates all this:

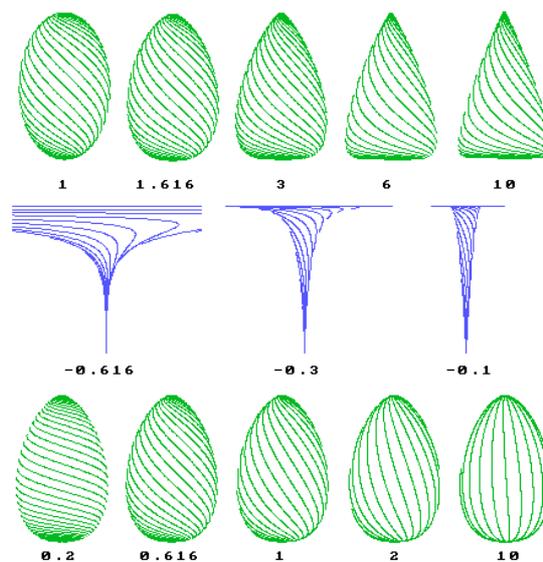


Figure 30

The top row shows a sequence of egg forms with increasing  $\lambda$  values as shown, with the elliptical form on the left, and on the right  $\lambda=10$ , already giving a nearly conical form. As  $\lambda$  increases towards infinity the form becomes a cone. The centre row shows three examples of different  $\lambda$  values for vortices. The bottom row shows a series of eggs all with the same  $\lambda$  value but with spirals that wind upwards ever more steeply. The numbers refer to another parameter  $\epsilon$  which relates to the steepness of the spirals, ranging from very small numbers for slowly ascending spirals towards infinity as the spirals tend towards vertical profiles. The steepness of the spirals depends upon the circling measures used in Figure 24: if they are the same then P moves round a circle at a constant height, and  $\epsilon = 0$ . As the measures become ever more distinct the spirals in the two fixed planes differ ever more giving ever steeper spirals. The parameter  $\epsilon$  is calculated from the relationship between the two measures. The point is that no matter how large or small, or how fat or thin the egg or vortex is, the parameters  $\lambda$  and  $\epsilon$  control the shape of the path curves. For many situations these two parameters suffice, which is a scientifically "strong" situation. A third parameter is required if we do not have symmetry about the vertical axis e.g. for a spiral surface:

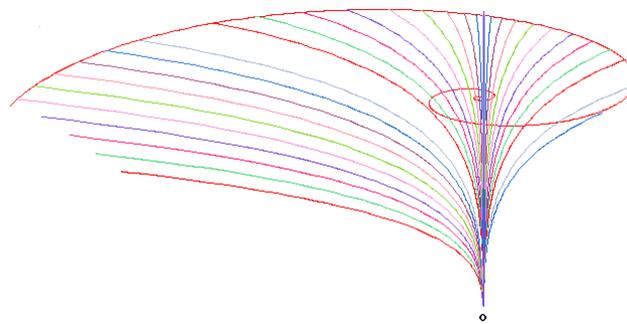


Figure 31

Here the horizontal cross sections are not circles as for eggs and vortices, but are logarithmic spirals. They could in fact be any path curve: eggs, spirals, circles, ellipses and so on.

The following diagram illustrates path curves where the top and bottom invariant planes in Figure 24 are not parallel:

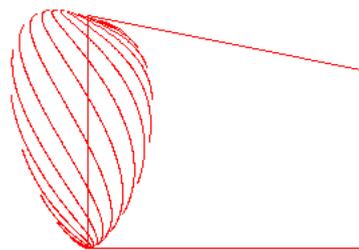


Figure 32

Lawrence Edwards found that such cases describe the left ventricle of the heart, relating to diastole when the planes are nearly parallel and systole when at their greatest angle.

## Research

Figure 26 suggests that path curves are related to natural forms as first suggested by George Adams (Reference 1). Lawrence Edwards realised that although this seems to be the case some serious research would be required to see whether such a relationship was fortuitous or whether the path curve process is actually occurring in Nature. There are two aspects to this question: the hypothesis that the geometry is related to natural processes, and if so to see how well its forms fit actual ones as in Figure 26. Thus the investigation of real forms is not merely a "fitting" exercise as has been claimed by critics, but rather a test of a hypothesis which we must first describe.

The hypothesis, following the research of Rudolf Steiner, is that another kind of space exists besides the one we are used to, and that laws related to both these spaces are active in Nature. The other kind of space is usually called *counterspace*, and is dual, or polar, to our ordinary space. Thus where points play a fundamental role in space, planes play that role in counterspace. A striking example of that concerns infinity. In ordinary space we feel that ideal points are infinitely far away outwards, whereas in counterspace planes are infinitely far away inwards. This is not what we experience with our waking day consciousness but rather is what can be experienced in a heightened consciousness which experiences "here" to be at the world periphery looking inwards from all directions, polar opposite or dual to our ordinary experience of being figuratively at a point in space looking outwards. Counterspace allows a genuine development of the holistic aspects of the world which are becoming increasingly apparent in modern physics, but raise paradoxes when handled in the traditional point-based fashion. George Adams (Reference 1) found a mathematical description of counterspace using the new developments in geometry in the 19th Century that have since proved so fruitful in various scientific contexts. This has been taken further by Conradt and Thomas (References 2, 7 and 8) as well as by other researchers.

George Adams envisaged that the two spaces interweave in path curves, and the way this is conceived was described above where a path curve is dually woven of points and planes. His idea was that subtle holistic forces work into living forms such as pine cones, flower buds, leaf buds, birds eggs and so on. Hence if these forms in Nature exhibit path curve forms accurately over a wide range of the parameters  $\lambda$  and  $\epsilon$  that is evidence in favour of the hypothesis. The variability of the forms is strictly limited by those parameters, so they cannot be fitted to just anything.

Lawrence Edwards conducted an extensive research programme to test the hypothesis, analysing thousands of bud forms in the process (Reference 3). He found some remarkably good results, within the variability of living forms subject as they are to wind and weather. All the flower buds he investigated showed unmistakable correlations with the path curve process. Below is one example by the author of how well these forms conform to the mathematics:

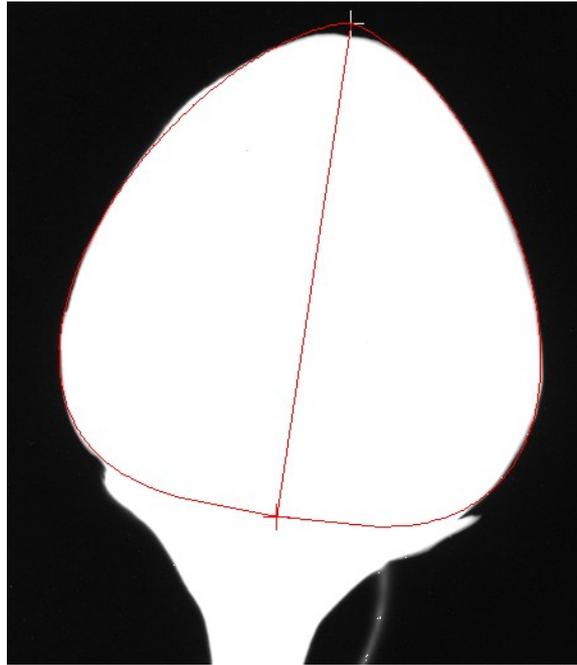


Figure 33

This shows an enlargement of a *Kerria Japonica* bud, which is itself only a few millimetres in size. The red curve shows the mathematical path curve calculated to give the best fit, which speaks for itself. In practice the deviation is calculated mathematically to make objective comparisons. In this case  $\lambda = 1.79$ . The sepals were removed which explains the uneven base.

Edwards started a certain line of investigation in 1983 which led to something quite unexpected (Reference 3). He measured the lambda values of leaf buds of trees during the winter, which are egg forms. The leaf buds had their  $\lambda$  value between 2 and 3. Edwards expected the leaf bud shapes to be static through the winter but to his surprise  $\lambda$  varied rhythmically up and down with a roughly two week cycle. As he looked into the matter further he found that different trees had slightly but distinctly different cycle lengths, and these correlated with the times between conjunctions and oppositions of the Moon and a planet so that for an oak, for example, this correlated closely with the conjunctions and oppositions with Mars. Over 19 years Edwards collected a large volume of data on this and found a method of analysis which confirms the relationship of different trees to different planets in this respect. Many examples of these rhythms can be seen in Reference 4.

Interestingly the effect was sharply reduced for a tree near a large transformer. Now it is scarcely possible in reasonable time to grow trees in special locations to test an idea! So he compared the behaviour of knapweed buds under electrical cables with others and confirmed this result. It is probably the frequency of the electrical or magnetic oscillation that interferes with a planar linkage between space and counterspace in the dynamic processes involved.

A further confirmation of the planetary influence came when the comet Shoemaker-Levy collided with Jupiter in 1994. Work with knapweed since 1985 had shown it to be connected with Jupiter, but it behaved quite uncharacteristically in 1994 (Reference 5). Usually the knapweed  $\lambda$  increases from about 1.3 at the end of June to slightly below 1.7 in mid August, and the results for hundreds of buds analysed between 1985 and 1993 were plotted to give a mean graph of the variation of  $\lambda$  in that July and August, together with two graphs of the extreme values of  $\lambda$  reached by individual buds either side of that mean.



Figure 34

In 1994 the whole graph lay well above the upper extreme with  $\lambda$  varying from just above 1.7 initially to over 1.8 and then gradually falling again after the impact, but still remaining near 1.8. Truly a remarkable result!

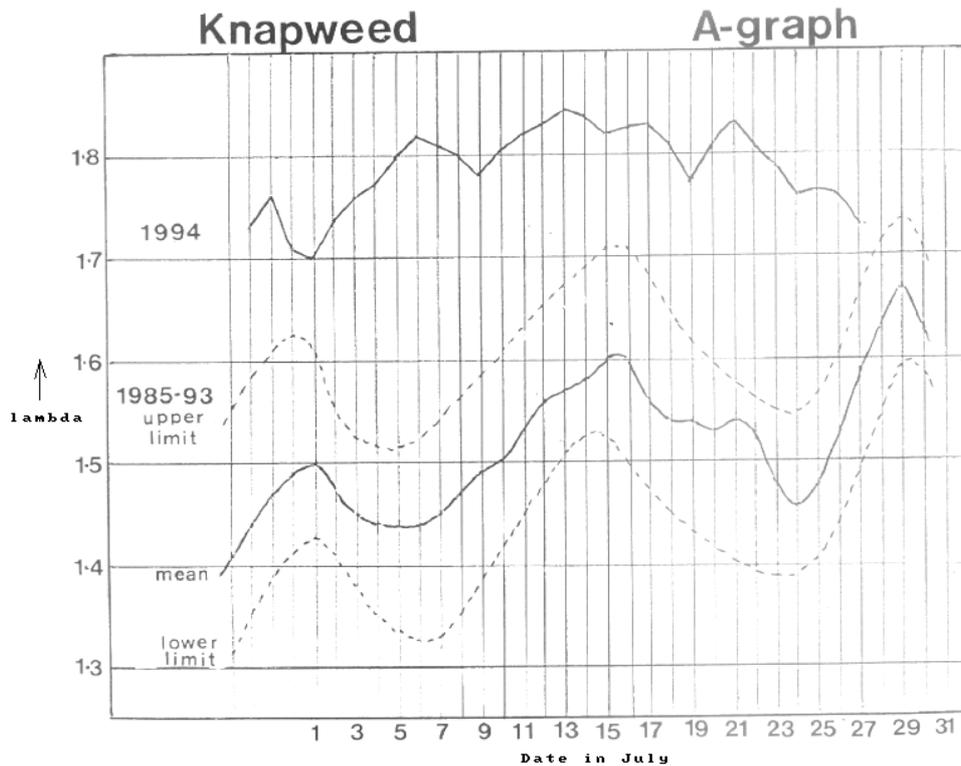


Figure 35

In 1983 the correlation between the rhythmic variation in bud shape and the conjunctions and oppositions of the Moon and planets seemed straight forward, but as the years passed there was a gradual 'slippage' so that although the period of the variations remained correlated the phase gradually changed so that several days between the maxima and minima of the graph and the conjunctions and oppositions began to accrue. After seven years the two were in step again and then the slippage continued as before. It has remained a mystery why this should happen and what governs the rate of slippage. A chart of the phase slippage constructed by Edwards is shown below.

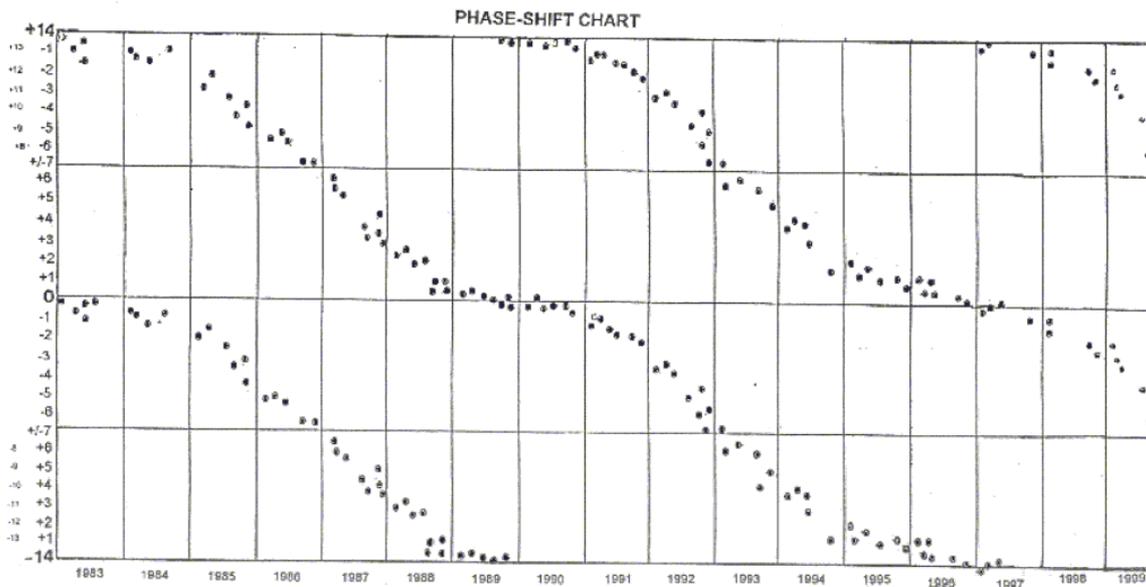


Figure 36

The possibility of cavities between space and countespace became apparent (Reference 8) and it seemed worth investigating whether there could be any connection. Now the size of a cavity varies in inverse proportion to the frequency involved, so that the lowest notes of an organ require the largest pipes and vice versa. A frequency of one cycle every two weeks is very low indeed yet the buds as cavities (if that they are) are very small. The scaling between space and countespace affects the frequency and when the scaling required to relate a bud-sized cavity to a two weekly rhythm was calculated it was very close to the reciprocal of the so-called velocity of light. It is nearly exact for a 4mm bud, and many buds are this sort of size, including leaf buds, although there is large variation more generally. The point is that rhythms in natural processes have been demonstrated to correlate with cosmic rhythms, and investigation of such connections rather than mere positional aspects of the planets is more likely to be fruitful. The work on resonant cavities bridging space and countespace provides a rationale for such research.

### Water Vortices

In Figure 13 we showed two types of vortex-like path curve. Lawrence Edwards showed that photographs taken by John Blackwood of stable water vortices fit the one on the right for a suitable value of  $\lambda$ . Since then Georg Sonder took this further and showed that a typical stable water vortex has  $\lambda = -2.9$  (Reference 6).

### Conclusion

The path curve process has been demonstrated in flower and leaf buds, birds eggs, pine cones, the human heart and the uterus during pregnancy. Such a wide range of application mostly using the single parameter  $\lambda$  is impressive. It is difficult to 'prove' that these forms 'are path curves' as that involves important philosophical questions about the nature of science. However as evidence for the process envisaged between space and countespace it is highly suggestive. In a further article another aspect of the work with pivot-transforms will be presented which firms up this evidence. A paradigm-shift away from exclusively local action to include genuine holism is called for, and references 2, 7 and 8 are a contribution to that.

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