# Geometry, Kinematics, and Rigid Body Mechanics in Cayley-Klein Geometries

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**Abstract.** The $19^{th}$ century witnessed a dramatic freeing of the human intellect from its naive assumptions. This development is particularly striking in mathematics. New geometries were discovered which challenged and expanded the familiar space of euclid, notably projective geometry and non-euclidean metric geometries. Developments in algebra led to similarly earth-shaking discoveries. This expansion of consciousness was accompanied by a desire to find a higher unity in the growing diversity. Hermann Grassmann made a bold, early attempt to bring the two domains together in his "Science of extension" of 1844. Later in the century, Cayley and Klein contributed to this process by their discovery that metric geometry could be constructed atop projective geometry. William Clifford introduced a *geometric* product combining the metric inner product with Grassmann's exterior product; the resulting structure has become known as a Clifford or geometric algebra. A completely satisfactory formulation of geometric algebra had to await the rigorous clarity of $20^{th}$ century mathematics.

The work presented here (the author's Ph. D. thesis from the Technical University Berlin, 2011) represents an attempt to bring this stream of $19^{th}$ century mathematics into the aforementioned rigorous, modern form, and to apply the resulting thought forms to understand kinematics and rigid body mechanics in the spaces under consideration (including euclidean, hyperbolic and elliptic space). One guiding light in the treatment is the reliance on a projective geometric foundation throughout, particularly a consequent application of the principle of duality. This projective approach guarantees that the result is *metric-neutral* – the results are stated in such a way that they apply equally well to the two non-euclidean geometries as well as euclidean geometry. And a consequent consideration of duality leads to the recognition that this classical triad must be extended by a fourth geometry, *dual euclidean* geometry.

Readers of the MPK will not be surprised to learn that dual euclidean geometry has a close affinity with counter-space [in German, *Gegenraum*], a concept which Rudolf Steiner introduced and developed, particularly in his natural science lectures. That such a connection exists in the work presented here is not surprising, as the initial impulse for this thesis was provided by the later works of George Adams (for example, see [Ada59]), to whom the present publication is dedicated. One of Adams' great gifts was to show how the results of modern science, particularly mathematical, when grasped with enlivened thinking, reveal deep, often unexpected, connections to the world and the human being – in a manner reminiscent of Goethe's reading the "open secrets" of nature. Such insights can indicate fruitful directions for further research. In this spirit is the current publication offered, with special thanks to Peter Gschwind for making it possible.

# Preface

This thesis arose out of a desire to understand and simulate rigid body motion in 2- and 3-dimensional spaces of constant curvature.The results are arranged in a theoretical part and a practical part. The theoretical part first constructs necessary tools – a family of real projective Clifford algebras – which represent the geometric relations within the above-mentioned spaces with remarkable fidelity. These tools are then applied to represent kinematics and rigid body dynamics in these spaces, yielding a complete description of rigid body motion there. The practical part describes simulation and visualization results based on this theory.

Historically, the contents of this work flow out of the stream of $19^{th}$ century mathematics due to Chasles, Möbius, Plücker, Klein, and others, which successfully applied new methods, mostly from projective geometry, to the problem of rigid body motion. The excellent historical monograph [Zie85] coined the name *geometric mechanics* expressly for this domain[1]. Its central concepts belong to the geometry of lines in three-dimensional projective space. The theoretical part of the thesis is devoted to formulating and occasionally extending these concepts in a modern, metric-neutral way using the real Clifford algebras mentioned above.

Autobiographically, the current work builds on previous work ([Gun93]) which explored visualization of three-dimensional manifolds modeled on one of these three constant curvature spaces. The dream of extending this geometric-visualization framework to include physics in these spaces – analogous to how in the past two decades the mainstream euclidean visualization environments have been gradually extended to include physically-based modeling – was a personal motivation for undertaking the research which led to this thesis.

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[1] although today there are other meanings for this term.

## Audience

The thesis is written with a variety of audiences in mind. In the foreground is the desire to present a rigourous, self-contained, metric-neutral elaboration of the mathematical results, both old and new. It is however not written for the specialist alone. For those interested only in the euclidean theory I have attempted to make it accessible to readers without background or interest in the non-euclidean thread. To be self-contained, many preliminary results from projective geometry and linear and multi-linear algebra are stated, with references to proofs in the literature. To be accessible, most of the results are stated and proved only for the dimensions $n = 2$ and $n = 3$, even when a general proof might present no extra difficulty. The exposition includes many examples, particularly euclidean ones, in which the reader can familiarize himself with the content. I have attempted at the ends of chapters to provide a guide to original literature for those interested in exploring further. Finally, as a firm believer in the value of pictures, I have tried to illustrate the text wherever possible.

## Outline

Chapter 1 introduces the important themes of the thesis via the well-known example of the Euler top, and shows how by generalizing the Euler top one is led to the topic of the thesis. In addition to reviewing the key ingredients of rigid body motion, it contrasts the historical approaches of Euler and Poinsot to the problem, and relates these to the approach taken here. It discusses the appropriate algebraic representation for the mathematical problems being considered. It shows how quaternions can be used to represent the Euler top, and specifies a set of properties which an algebraic structure should possess in order to serve the same purpose for the extended challenge posed by the thesis.

Chapter 2 introduces the non-metric foundations of the thesis. The geometric foundation is provided by real projective geometry. From this is constructed the Grassmann, or exterior, algebra, of projective space. A distinction is drawn between the Grassmann algebra and its dual algebra; the latter plays a more important role in this thesis than the former. We discuss Poincaré duality, which yields an algebra isomorphism between these two algebras, allowing access to the exterior product of the one algebra within the dual algebra without any metric assumptions.

Chapter 3 introduces the mathematical prerequisites for metric geometry. This begins with a discussion of quadratic forms in a real vector space $\mathsf{V}$ and associated quadric surfaces in projective space $\mathbf{P}(\mathsf{V})$. A class of *admissable* quadric surfaces are identified – which include non-degenerate and “slightly” degenerate quadric surfaces – which form the focus of the the subsequent development.

Chapter 4 begins with descriptions of how to construct the elliptic, hyperbolic, and euclidean planes using a quadric surface in $\mathbb{R}P^2$ (also known as a conic section in this case), before turning to a more general discussion of Cayley-Klein spaces and Cayley-Klein geometries. We establish results on Cayley-Klein spaces based on the admissable quadric surfaces of Chapter 3 – which are amenable to the techniques described in the rest of the thesis.

In Chapter 5 the results of the preceding chapters are applied to the construction of real Clifford algebras, combining the outer product of the Grassmann algebra with the inner product of the Cayley-Klein space. We show that for Cayley-Klein spaces with admissable quadric surfaces, this combination can be successfully carried out. For the 3 Cayley-Klein geometries in our focus, we are led to base this construction on the *dual* Grassmann algebras. We discuss selected results on $n-$dimensional Clifford algebras before turning to the 2- and 3-dimensional cases.

Chapter 6 investigates in detail the use of the Clifford algebra structures from Chapter 5 to model the metric planes of euclidean, elliptic, and hyperbolic geometry.[2] The geometric product is exhaustively analyzed in all it variants. Following this are metric-specific discussions for each of the three planes. The implementation of direct isometries via conjugation operators with special algebra elements known as rotors is then discussed, and a process for finding the logarithm of any rotor is demonstrated. A typology of these rotors into 6 classes is introduced based on their fixed point sets.

Taking advantage of the results of Chapter 6 wherever it can, Chapter 7 sets its focus on the role of non-simple bivectors, a phenomenon not present in 2D, and one which plays a pervasive role in the 3D theory. This is introduced with a review of the line geometry of $\mathbb{R}P^3$, translated into the language of Clifford algebras used here. Classical results on line complexes and null polarities – both equivalent to bivectors – are included. The geometric products involving bivectors are exhaustively analyzed. Then, the important 2-dimensional subalgebra consisting of scalars and pseudoscalars is discussed and function analysis based on it is discussed. Finally, the *axis* of a rotor is introduced and explored in detail. These tools are then applied to solve for the logarithm of a rotor in the 3D case also. We discuss the exceptional isometries of Clifford translations (in elliptic space) and euclidean translations in detail. Finally, we close with a discussion of the continuous interpolation of a metric polarity. We demonstrate a solution which illustrates the power and flexibility of these Clifford algebra to deal with challenging geometric problems.

Having established and explored the basic tools for metric geometry provided by these algebras, Chapter 8 turns to kinematics. The basic object is an isometric motion: a continuous path in the rotor group beginning at the identity. Taking derivatives in this Lie group leads us to the Lie algebra of bivectors. The results of Chapter 7 allow us to translate familiar results of Lie theory into this setting with a minimum of machinery. We analyse the vector field associated to a bivector, considered as an instantaneous velocity state. In deriving a transformation law for different coordinate systems we are led to the

[2] The decision to begin with the 2D case rather directly with the 3D was based on the conviction that this path offers significant pedagogical advantages due to the unfamiliarity of many of the underlying concepts.

Lie bracket, in the form of the commutator product of bivectors. Finally, for noneuclidean metrics, we discuss the dual formulation of kinematics in which the role of point and plane, and of rotation and translation, are reversed.

The final theoretical chapter, Chapter 9, treats rigid body dynamics in the 3D setting. This begins with a metric-neutral treatment of statics. Movement appears via newtonian particles, whose velocity and impulse are defined in a metric-neutral way purely in terms of bivectors and the metric quadric. Rigid bodies are introduced as collections of such particles. The inertia tensor is defined and shown to be a symmetric bilinear form on the space of bivectors. We introduce a second Clifford algebra, on the space of bivectors, whose inner product is derived from this inertia tensor. We derive Euler equations for rigid body motion, and indicate how to solve them. Finally, we discuss the role of external forces and discuss work and power in this context.

Chapter 10 provides a brief introduction to dual euclidean geometry and its associated Clifford algebra. It begins by showing that the set of four geometries: euclidean, dual euclidean, elliptic and hyperbolic, form an unified family closed under dualization. It compares the dual euclidean plane to the euclidean plane via some elementary examples, and indicates some interesting research possibilities for this geometry.

Once these theoretical results have been established, experimental results based on this theory are presented in Chapter 11. The focus is on the noneuclidean spaces, with the euclidean results being mainly useful as quality control. First the two-dimensional case is handled. A variety of qualitative behaviors are presented and discussed with reference to the theoretical results already presented. Then the three-dimensional case is handled, and some behaviors not seen in the 2D case are shown and discussed. The presentation of these results is accompanied by a description of visualization strategies and tools developed to assist in the presentation and analysis of the results, in both 2D and 3D.

The concluding chapter, Chapter 12, reflects on the results presented and provides an overview of innovative aspects, ranging from concrete to methodological.

## Acknowledgements

This work owes its existence to many people, a few of whom I want to thank by name. I begin by gratefully acknowledging my debt to George Adams (1894-1963), whose work on dual euclidean geometry and its connections to modern science provided me with the impulse to undertake this work, and whose qualitative approach to mathematics remains a constant source of inspiration for me. I would like to thank my parents, Charles and Virginia† Gunn, who encouraged me to develop my inborn interests and supported me in countless ways, Thomas Brylawski, Ph. D., (1944-2007), Professor of Mathematics at the University of North Carolina at Chapel Hill, was the advisor of my master's project there and tireless encourager of my slumbering capacities. I want to thank Bill Thurston (and colleagues at the Geometry Center in the years 1987-1993) for sharing his 3D geometric magic with me particularly his mastery of 3D noneuclidean geometry. Thanks also are due my thesis advisor, Ulrich Pinkall, for his expert and tolerant guidance. Also, without support from the DFG Research Center Matheon in the academic year 2010-2011, the thesis would not have been possible in its present form. Special thanks to my wife Edeltraud and daughter Lucia, who have patiently accompanied me on the odyssey of this thesis over the past 8 years.

Finally, thanks to all the mathematicians living and dead whose works flowed into this project. Most of the better parts of this thesis are certainly due to their impulses; the parts due to me will have served their purpose if they bring these impulses a step further in a scientific sense and a step wider, to a larger audience.

Berlin, August 2011 (revised March, 2014)

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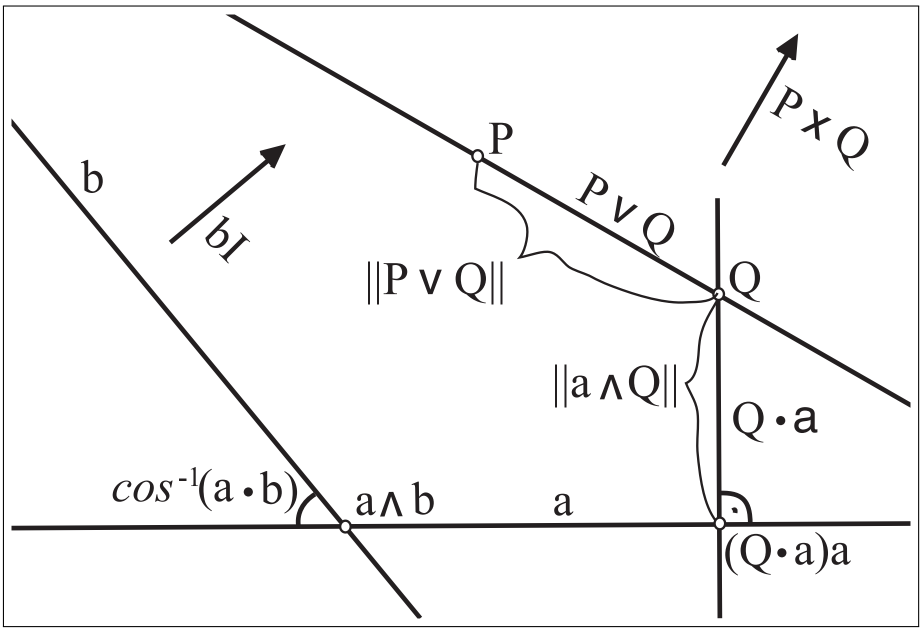


**Preview sample image.** Figure 6.2: A graphical representation of selected geometric products between various $k$-blades in the euclidean algebra $Cl_0^2$. Points and lines are assumed to be normalized. Ideal points are drawn as vectors, distances indicated by norms.

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# Chapter 1
# Preview: the Euler Top

One of the main goals of this thesis, as sketched in the Forward, is to provide a modern understanding of rigid body motion in the 3-dimensional non-euclidean spaces of elliptic and hyperbolic geometry. A natural starting point for this investigation is provided by the well-known example of the *Euler top*, one of the simplest non-trivial examples of rigid body motion. This is a rigid body in three-dimensional euclidean space, constrained to move around its center of mass, and not subject to any external forces. Not only does this example serve to identify the key components of the analysis of rigid body motion; we will show below (Sect. 9.5.2) that the content of this thesis is a natural extension of the Euler top when one removes the constraint that the motion has a fixed point. Furthermore, the differing approaches to the Euler top represented by Euler and Poinsot, also throws an important light on the choice of methods adopted in this thesis.

The discussion here is not intended to be mathematically rigorous. Readers for which the material is unfamiliar are encouraged to consult standard literature on rigid body mechanics such as [Arn78], Ch. 2. Proofs for all the results presented here can also be obtained from the thesis itself by re-introducing the constraint that the center of mass is fixed by the motion. See Section Sect. 9.5.2 for details. A much fuller account of the historical details presented here can be found in [Zie85].

## 1.1 Euler and the analytic approach

The question of rigid body mechanics entered the mathematical literature with the investigations of Euler and d'Alembert (working separately around 1760) [Zie85]. The problem which they solved was: given the mass distribution of the rigid body and its initial velocity, to find a path in the isometry group $SO(3)$ of rotations of $\mathbb{R}^3$, which describes the position of the body at each subsequent moment of time. Each obtained a complete description of the motion of an Euler top as the solution of a set of ordinary differential equations. The differential equations for the instantaneous velocity of the object became known as *Euler equations* of the motion. Lagrange (1788) introduced a more abstract setting in which the motion of the Euler top could be solved.

The key feature of all the analytic approaches is that attention is focused on the isometry group of the rigid body rather than on the ambient space of the rigid body.[1] This is easy to overlook since the dimension of both spaces in the case of the Euler top is 3. As a result, the analytic solution does not immediately provide any detailed description of how the motion proceeds within the ambient space of the body.

### 1.1.1 Poinsot and the geometric approach

This unsatisfactory state of affairs was addressed and remedied by Poinsot in [Poi51] (English translation [Poi84]), based on a work first presented in 1834. This work, made possible by the dramatic developments in geometry at the turn of the $19^{th}$ century notably in the school led by Monge (1746-1818), is based on a geometric approach, in contrast to the analytic approach pioneered by Euler and Lagrange. Poinsot first describes his dissatisfaction with the results of the analytic approach:

> ...it must be allowed, that in all these solutions [of Euler, d'Alembert, and Lagrange], we see nothing but calculations, without having any clear idea of the rotation of the body. We may be able by calculations, more or less long and complicated, to determine the place of the body at the end of a given time; but we do not see at all how it arrives there. [[Poi84], p. 2]

and goes on to describe his alternative approach and its advantages:

> Therefore to furnish a clear idea of this rotatory motion, hitherto unrepresented, has been the object of my endeavors. The result is an entirely new solution to the problem of [the Euler top]: a genuine solution, inasmuch as it is palpable, and enables us to follow the motion of the body as clearly as the motion of a point. And if we would pass from this geometrical representation to calculation ... the formulae required for the purpose are direct and simple, each of them expressing a dynamical theorem of which we have a clear idea, and which proceeds at once to its object. [[Poi84], p. 3]

Finally, Poinsot reflects on the success of his method as being a result of a particular penetration of the phenomena with exactly the correct mathematical concepts:

> For we may remark generally of our mathematical researches, that these auxiliary quantities, these long and difficult calculations into which we are often drawn, are almost always proofs that we have not in the beginning considered the objects themselves so thoroughly and directly as their nature requires, since all is abridged and simplified, as soon as we place ourselves in a right point of view. [[Poi84], p. 4]

[1] This is a modern formulation; the group concept had not yet been introduced in the $18^{th}$ century.

From this we can infer that the distinction between his method and that of his predecessors is not only *geometric vs. analytic*; it is just as much *concrete vs. abstract*. Poinsot's achievement is based on his focusing on the concrete conditions of the Euler top, and out of these concrete conditions, deriving a description that avoids the complexities inherent in the more abstract approach of Euler and Lagrange. His method is more comprehensive than the analytic one, since using it he was able to derive all the results obtained by Euler and Lagrange for the Euler top, but the opposite is not true.

## 1.2 Ingredients of the motion of the Euler top

Before proceeding further we provide a quick review of the ingredients of these solutions of the Euler top. Fig. 1.1 shows a diagramatic representation of an Euler top at a particular instant of its motion. In this case the rigid body is represented by the yellow wireframe box, and consists of 8 particles positioned at the corners of the box. That it is a *rigid* body means that the distances of all particle pairs remains fixed under the motion. The other elements in the figure will be discussed in the subsequent discussion of Poinsot's contributions.

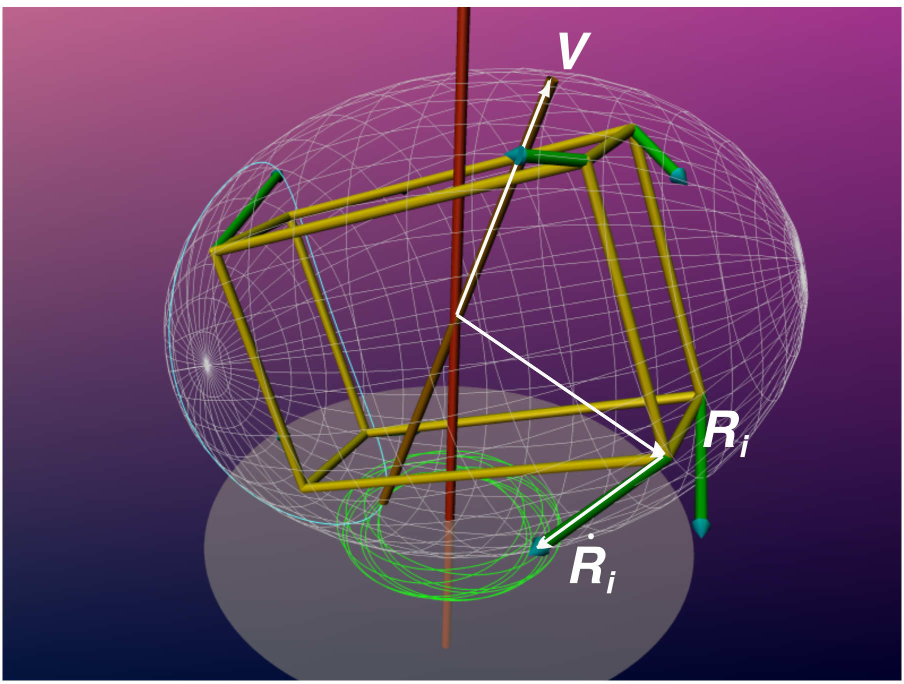


**Fig. 1.1** The angular velocity $\mathbf{V}$ determines the linear velocity $\dot{\mathbf{R}}_i$ of the particle of the rigid body located at position $\mathbf{R}_i$.

The instantaneous motion of an Euler top is a rotation around an axis passing through the fixed point. Such an element is called an *angular velocity* and is represented by the bronze axis labeled $\mathbf{V}$. This angular velocity imparts to each particle $\mathbf{R}_i$ a direction and intensity of motion represented by the vector $\dot{\mathbf{R}}_i := \mathbf{V} \times \mathbf{R}_i$. The angular momentum of the particle is then obtained by $\mathbf{M}_i := m_i \mathbf{R} \times \dot{\mathbf{R}}_i$ where $m_i$ is the mass of the particle at $\mathbf{R}_i$. One also defines the kinetic energy of the particle as $E_i := \frac{m}{2}\|\dot{\mathbf{R}}_i\|^2$. The absence of external forces implies that both $\mathbf{M}_i$ as well as $E_i$ is a conserved quantity.

When one sums over all the particles in the body, one obtains aggregate momentum and kinetic energy for the body:

$$\mathbf{M} = \sum_i \mathbf{M}_i, \qquad E := \sum_i E_i$$

These are naturally also conserved quantities.

Expanding out the summation for the energy yields an expression which depends quadratically on the angular velocity $\mathbf{V}$. One can express this dependence in a symmetric bilinear form $\mathbf{A}$, the *inertia tensor* of the body, and arrive at the formula $E = \mathbf{A}(\mathbf{V}, \mathbf{V})$. This in turn provides a similar form for the momentum: $\mathbf{M} = \mathbf{A}(\mathbf{V})$. Here, the occurrence of $\mathbf{A}$ represents the *polarizing* operator associated to a symmetric bilinear form. This shows that the momentum is a dual vector with respect to the angular velocity.

By considering the fact that the momentum is conserved, one can derive the Euler equations for the angular velocity in the body:

$$\dot{\mathbf{V}}_c = \mathbf{A}^{-1}(\mathbf{V}_c \times \mathbf{M}_c) \tag{1.1}$$

To obtain the motion of the rigid body as a path $\mathbf{g}$ in the Lie group $SO(3)$, the rotations of $\mathbb{R}^3$, one can then integrate $\dot{\mathbf{g}}$ using the relation $\dot{\mathbf{g}} = \mathbf{g}\mathbf{V}_c$.

### 1.2.1 Poinsot description

The description above essentially reflects the thought-process of the Euler approach to the Euler top. [Poi51] provides a much fuller geometric description of this motion. The elements of this description are shown in Fig. 1.2. The angular velocity is assumed to be given. Then the angular momentum, defined as above by $\mathbf{M} = \mathbf{A}(\mathbf{V})$ is an element of the dual space, hence a plane; it is traditionally represented by the normal direction of this plane, in this case, the red vertical axis. For a given choice of $\mathbf{M}$, the set of all angular velocity $\mathbf{V}$ which yield the same kinetic energy $E$ is a quadric surface called by Poinsot the *inertia ellipsoid.* It is shown as a white wireframe ellipsoid in the figure.

Poinsot provides a geometric understanding of how the angular velocity and angular momentum evolve in $\mathbb{R}^3$. The path of the angular velocity vector, as the rigid body moves, is called by Poinsot the *polhode* of the motion. Since the energy $E$ is conserved, the polhode is constrained to lie on the surface of the inertia ellipsoid. On the other hand, conservation of the momentum

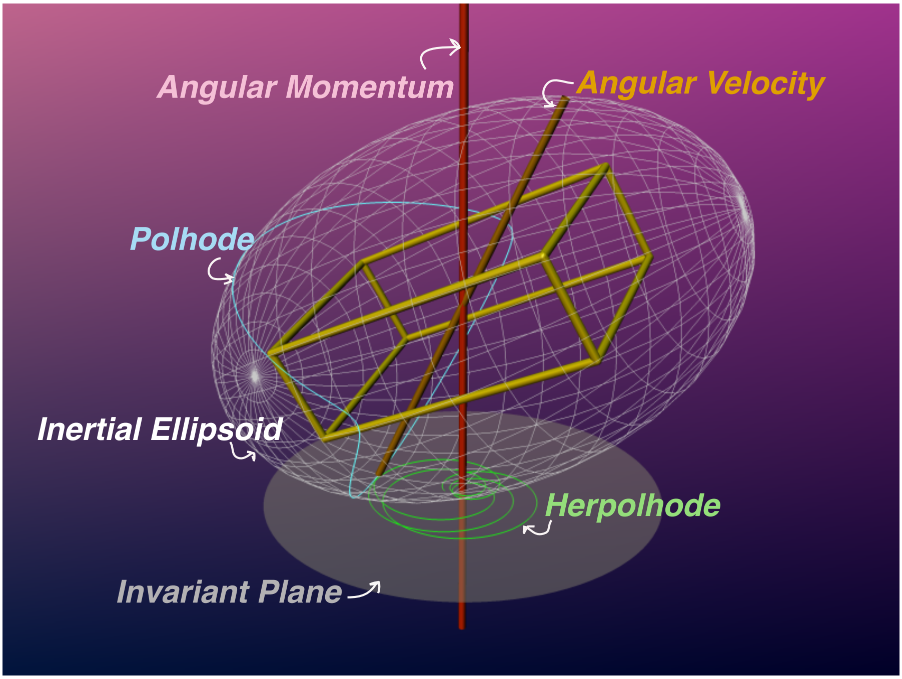


**Fig. 1.2** The Poinsot description of the motion provides geometric interpretation to the elements of the analytic description of the motion.

vector in space implies conservation of its length in body coordinate system: $\|\mathbf{A}(\mathbf{V})\| = k$, which represents another, confocal ellipsoid. Hence the polhode is the intersection of these two ellipsoids, a closed quartic curve on the inertia ellipsoid. It is the cyan curve in the figure.

In the world coordinate system, where the momentum is fixed, the condition that $\langle \mathbf{V}, \mathbf{M} \rangle = E$ represents a plane perpendicular to the angular momentum, called by Poinsot the *invariant plane*, which appears in gray at the bottom of the figure. The green curve in the invariant plane is the path of the angular velocity in the world coordinate system, and was called by Poinsot the *herpolhode*. As the body moves, the angular velocity vector traces out the polhode on the inertia ellipsoid and the herpolhode on the invariant plane. This means that the inertia ellipsoid rolls on the invariant plane during the motion, its point of contact being the current angular velocity. The herpolhode is a quasi-periodic curve that, generically, fills in an annulus of the invariant plane, where the inner (outer) boundary circle of the annulus corresponds to angular velocities with minimum (maximum) speed.

### 1.2.2 Generalizing the Euler top

One obtains the theme of this thesis if one replaces the condition that the body has a fixed point, with the condition that the body is free to move in a (3-dimensional) space of constant curvature. The details of this claim are established in the introductory chapters of the thesis, where it is shown that there are three such spaces – euclidean, elliptic, and hyperbolic space. Furthermore, the isometry groups of these spaces are all 6-dimensional Lie groups which contain $SO(3)$ as a subgroup.

[Arn78], Appendix 2, provides a methodology to deduce and solve the Euler equations for rigid body motion in an abstract setting which includes the three spaces above. Arnold's approach works with any Lie group; the role of the inertia tensor is taken over by a left-invariant metric on the corresponding Lie algebra. That is, if one has an inertia tensor, one obtains such a left-invariant metric; but one can in fact work with the wider class of left-invariant metrics and obtain and solve ODE's. This is a useful approach for a first solution.

However, all the objections raised by Poinsot to the analytic approach apply here to the Arnold approach. All the calculations take place in the 6-dimensional spaces of the Lie group and the Lie algebra. There is, *a priori*, no geometric insight into how the motion unfolds in the underlying 3-dimensional space where the rigid body and the observer are at home. Due to this limitation, the current thesis adopts the attitude of Poinsot, and sets the goal of providing a geometric description of the rigid body motion which as much as possible refers to geometric entities in the underlying 3-dimensional space where the motion occurs.

## 1.3 Algebraic representation

The goal of providing a Poinsot description of rigid body motion brings with it the question of what mathematical representation is best-fitted to achieve that goal. The historical work of Euler and Poinsot preceded the development of modern algebra. The majority of the current literature on rigid body motion employs linear algebra to represent the isometry groups and their action on the points of the ambient space of the body. The current work departs from that trend in its use of geometric (or Clifford) algebra for that purpose. The best way to motivate this choice is to return to the example of the Euler top and show how *quaternions* can be profitably used to model the rigid body motion. Then, after this excursion, we discuss the ways this algebraic structure needs to be extended to handle the spaces considered by this thesis.

### 1.3.1 Quaternions

William Rowan Hamilton discovered quaternions in 1843. Our aim here is not to provide an exhaustive account of quaternions, but just to present enough results to indicate the direction followed in the sequel.

Recall some facts about quaternions. Begin with $\mathbb{R}^4$ with basis $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$. Introduce a product structure on the basis elements:

$$\mathbf{1}^2 = 1; \quad \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$$
$$\mathbf{1} \text{ commutes with } \mathbf{i}, \mathbf{j}, \text{ and } \mathbf{k}.$$
$$\mathbf{ij} = -\mathbf{ji}, \quad \mathbf{jk} = -\mathbf{kj}, \quad \mathbf{ki} = -\mathbf{ik}$$

Extend this product by linearity to all of $\mathbb{R}^4$. This yields an associative, non-commutative algebra called the quaternions, written $\mathbb{H}$. $\mathbf{1}$ is the identity element.

**Definition 1.** For a quaternion $\mathbf{a} := a_0\mathbf{1} + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$:

- $\mathbf{a}_s := a_0\mathbf{1}$ is the *scalar* part of $\mathbf{a}$.
- $\mathbf{a}_v := a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ is the *vector* part of $\mathbf{a}$.
- If $\mathbf{a} = \mathbf{a}_v$, $\mathbf{a}$ is an *imaginary* quaternion; the set of imaginary quaternions is denoted $\mathbb{IH}$.
- For $\mathbf{a} = \mathbf{a}_s + \mathbf{a}_v$, $\overline{\mathbf{a}} := \mathbf{a}_s - \mathbf{a}_v$ is called the *conjugate* of $\mathbf{a}$.
- $\mathbf{a} \cdot \mathbf{b} = \mathbf{a}\overline{\mathbf{b}}$ is the *inner product* of $\mathbf{a}$ and $\mathbf{b}$.
- For non-zero $\mathbf{a}$, $\mathbf{a}^{-1} := \dfrac{\overline{\mathbf{a}}}{\mathbf{a} \cdot \mathbf{a}}$ is the *inverse* of $\mathbf{a}$.
- $\|\mathbf{a}\| := \sqrt{\mathbf{a}\overline{\mathbf{a}}}$ is the *norm* of $\mathbf{a}$.
- If $\|\mathbf{a}\| = 1$, $\mathbf{a}$ is a *unit* quaternion.

*Remark 2.* Verify that the definitions make sense. $\mathbf{a} \cdot \mathbf{b} \in \mathbb{R}\mathbf{1}$. The inverse satisfies $\mathbf{a}^{-1}\mathbf{a} = \mathbf{a}\mathbf{a}^{-1} = 1$.

The set of unit quaternions can be identified with the 3-dimensional sphere $\mathbf{S}^3$. We identify $\mathbb{IH}$ with $\mathbb{R}^3$ in the obvious way.

Let $\cdot$ and $\times$ be the inner and cross products, resp., on $\mathbb{R}^3$. Then for $\mathbf{g}, \mathbf{h} \in \mathbb{IH}$ one can verify directly that:

$$\mathbf{gh} = -\mathbf{g} \cdot \mathbf{h} + \mathbf{g} \times \mathbf{h}$$

Thus, the quaternion product combines the inner product with the cross product of $\mathbb{R}^3$.

A unit quaternion $\mathbf{g}$ can be written as $\cos\theta + \sin\theta(\mathbf{u})$ where $\mathbf{u}$ is a unit imaginary quaternion satisfying $\mathbf{u}^2 = -1$. Then evaluate the exponential function as a power series to obtain:

$$\mathbf{g} = e^{\theta\mathbf{u}}$$

For $\theta \in [0, 2\pi)$, this is a bijective mapping $\mathbb{IH} \leftrightarrow \mathbf{S}^3$.

Consider the product $\underline{\mathbf{g}}(\mathbf{x}) := \mathbf{g}\mathbf{x}\mathbf{g}^{-1} = \mathbf{g}\mathbf{x}\overline{\mathbf{g}}$ for unit quaternion $\mathbf{g}$ and imaginary quaternion $\mathbf{x}$. Write $\mathbf{g} = \cos\theta + \sin\theta(\mathbf{u})$. Then one can show that

$\underline{\mathbf{g}}$ is a rotation of $\mathbb{R}^3$ around the axis $\mathbf{u}$ of an angle $2\theta$. Under this mapping, $\underline{\mathbf{g}}$ and $\underline{-\mathbf{g}}$ give the same rotation. Hence, one obtains a map:

$$\mathbb{IH} \xrightarrow{e\ (1:1)} \mathbf{S}^3 \xrightarrow{\underline{\mathbf{g}}\ (2:1)} SO(3)$$

We review the prominent features of the configuration described above:

I. The quaternion product $\mathbf{gh}$ combines a symmetric and an anti-symmetric part.
II. $\mathbb{IH}$ is a vector subspace of $\mathbb{H}$, and $\mathbf{S}^3$ is a sub-group of $\mathbb{H}$ such the exponential map $e : \mathbb{IH} \to \mathbf{S}^3$ is locally a bijection and globally a covering map. The nice properties of this map depend on the fact that the elements of $\mathbb{IH}$ have scalar square.
III. The map $\mathbf{S}^3 \to SO(3) : \mathbf{g} \to \underline{\mathbf{g}}$ is a 2:1 covering of the rotation group of $\mathbb{R}^3$.

### 1.3.2 The Euler top via quaternions

One can represent elements of $SO(3)$ and its Lie algebra via quaternions. The motion $\mathbf{g}$ becomes a path in $\mathbf{S}^3$; the angular velocity and momentum become elements of $\mathbb{IH}$. The Euler equations become:

$$\begin{aligned} \dot{\mathbf{g}} &= \mathbf{g}\mathbf{V}_c \\ \dot{\mathbf{M}}_c &= \frac{1}{2}(\mathbf{V}_c\mathbf{M}_c - \mathbf{M}_c\mathbf{V}_c) \end{aligned}$$

Here all products are the quaternion product. We mention two advantages of the quaternion approach:

1. The representation of isometries is *geometric*: the axis of a rotation $\mathbf{r} \in \mathbf{S}^3$ is the imaginary part of $\mathbf{r}$.
2. The representation is *compact*: an isometry is represented by 4 real numbers. Compare this to the matrix approach, where 9 real numbers are required. This compactness has significant advantages in numerical applications, for example, in solving differential equations, since one has many fewer directions of moving away from the correct solution.

### 1.3.3 Quaternion-like algebras for spaces of constant curvature

Motivated by these advantages, the current thesis incorporates algebras, analogous to the quaternions, corresponding to the larger isometry groups of the spaces under investigation. It turns out to be possible to find algebras which not only fulfill properties analogous to I, II, and III above, but which possess further attractive properties.

These algebras are obtained by introducing a graded algebraic structure in which different grades represent different-dimensional subspaces. Such a graded algebra is called a *Grassmann algebra*. The next step involves adding an inner product, which reflects the underlying metric properties of the space, to yield a *Clifford algebra*. The full power of this approach to represent a variety of interesting spaces is only enabled when one works within projective space rather than vector space. These are the mathematical foundations which form the next four chapters of this study. There follow two chapters showing how to use these algebras to do geometry in 2- and 3-dimensional spaces of constant curvature. These tools then provide the basis for investigating kinematics and rigid body mechanics in these spaces (Chapter 8 and Chapter 9).

# Chapter 2
# Projective foundations

This chapter reviews the non-metric mathematical structures – projective space and exterior algebra – required for the rest of the thesis. The choice of results presented here is conditioned by the requirements of later chapters. Consequently, particular attention is paid to establishing the principle of duality.

## 2.1 Projective geometry

**Real projective n-space** Let $\mathsf{V}$ be a real vector space of dimension $(n+1)$, and $\mathsf{V}^*$ its dual space. Let $\langle \mathbf{u}, \mathbf{x} \rangle = \mathbf{u}(\mathbf{x})$ represent the scalar product on $\mathsf{V} \otimes \mathsf{V}^*$ given by the evaluation map of a dual vector (linear functional) applied to a vector.

Then the n-dimensional projective space $\mathbf{P}(\mathsf{V})$ is obtained from $\mathsf{V}$ by introducing an equivalence relation on vectors $\mathbf{x}, \mathbf{y} \in \mathsf{V} \setminus \{\mathbf{0}\}$ defined by: $\mathbf{x} \sim \mathbf{y} \iff \mathbf{x} = \lambda \mathbf{y}$ for some $\lambda \neq 0$. That is, points in $\mathbf{P}(\mathsf{V})$ correspond to lines through the origin in $\mathsf{V}$. We sometimes write this equivalence relation $\mathbf{x} \equiv \mathbf{y}$ if two vectors represent the same projective point. See also Sect. 2.4 below.

*Remark 3.* For most of this work, $\mathsf{V} = \mathbb{R}^{n+1}$ (or $(\mathbb{R}^{n+1})^*$) and $\mathbf{P}(\mathsf{V}) = \mathbb{R}P^n$ (or $(\mathbb{R}P^n)^*$), real projective space of dimensions $n$ (or its dual). However, note that in many contexts it is not considered as an *inner product* space, that is, we do not assume it is equipped with an inner product. This differentiation will become more clear in Chapter 3 where metrics are introduced.

### 2.1.1 Projectivities

We review some facts about projective transformations which will be important in Chapter 5 since they provide the basis of the theory of isometries for the metric spaces under consideration.

**Definition 4.** Given four points $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{R}P^1$ with homogeneous coordinates $\mathbf{a} = (a_0, a_1)$, etc.. The *cross ratio* of the four points, written $(\mathbf{a}, \mathbf{b}; \mathbf{c}, \mathbf{d}) := \dfrac{|\mathbf{a}, \mathbf{c}||\mathbf{a}, \mathbf{d}|}{|\mathbf{b}, \mathbf{c}||\mathbf{b}, \mathbf{d}|}$, where .$|\mathbf{a}, \mathbf{c}|$ denotes the determinant $a_0 c_1 - a_1 c_0$, etc.

**Definition 5.** A *projectivity* of $\mathbb{R}P^1$ is a bijective map $\mathbb{R}P^1 \to \mathbb{R}P^1$ which preserves the cross ratio.

To obtain a similar notion for higher dimensions, we introduce an alternative definition:

**Definition 6.** For $n > 1$, a *projectivity* is a bijective map $\mathbb{R}P^n \to \mathbb{R}P^n$ or $\mathbb{R}P^n \to (\mathbb{R}P^n)^*$ which preserves linear dependence, and linear independence, of sets. The former is called a *collineation*, the latter, a *correlation*.

From this definition it is possible to deduce the following two theorems.

**Theorem 7.** *A projectivity is uniquely determined by its action on a linearly independent set of $n + 2$ points.*

*Remark 8.* Typically, these points are provided by $n + 1$ basis vectors $\mathbf{e}_i$ and the so-called *unit* point $\mathbf{u}$. For our purposes, we choose the unit point to be $\mathbf{u} := \sum_i \mathbf{e}_i$.

**Theorem 9.** *A projectivity preserves the cross ratio of 4 collinear points.*

For a proof, see [Spe63], §21.

The collineations form a group. This group is generated by a set of involutions described as follows.

**Definition 10.** Let $\mathbf{Z}$ be a point and $\mathbf{m}$ be a hyperplane in $\mathbb{R}P^n$ such that $\mathbf{Z}$ is not incident with $\mathbf{m}$. Then the *harmonic homology* with center $\mathbf{Z}$ and axis $\mathbf{m}$ is the collineation $H_{Z,m}$ defined by:

$$H_{Z,m}(\mathbf{P}) = -\langle \mathbf{Z}, \mathbf{m} \rangle \mathbf{P} + 2\langle \mathbf{P}, \mathbf{m} \rangle \mathbf{Z} \tag{2.1}$$

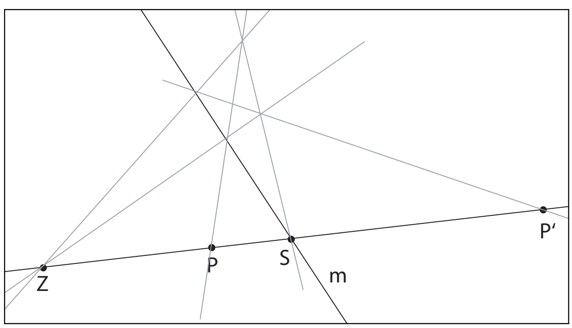


**Fig. 2.1** A harmonic homology with center $\mathbf{Z}$ and axis $\mathbf{m}$ acting on a point $\mathbf{P}$. In gray, a harmonic quadrilateral determined by $\mathbf{P}, \mathbf{Z}$, and $\mathbf{S}$, which determines $\mathbf{P'}$ as "hamonic fourth point" to other three points. Other choices of this cross ratio $\lambda$ lead a centered collineation with factor $\lambda$.

A harmonic homology fixes the center (linewise) and the axis (pointwise), and its action on a point $\mathbf{P}$ is as follows: find the intersection $\mathbf{S}$ of the line $\mathbf{k}$ joining $\mathbf{Z}$ with $\mathbf{P}$, with the axis $\mathbf{m}$. Then $\mathbf{P}' := H_{Z,m}(\mathbf{P})$ is the unique point of $\mathbf{k}$ such that the point pairs $(\mathbf{Z}, \mathbf{S})$, $(\mathbf{P}, \mathbf{P}')$ separate each other harmonically. See Fig. 2.1. For a proof see the discussion below of centered collineations.

*Remark 11.* The harmonic homology is a special case of a *centered collineation*, a collineation of the form:

$$P_{Z,m,\lambda}(\mathbf{P}) = \lambda\langle \mathbf{Z}, \mathbf{m}\rangle \mathbf{P} + (1-\lambda)\langle \mathbf{P}, \mathbf{m}\rangle \mathbf{Z} \tag{2.2}$$

A centered collineation has center $\mathbf{Z}$ and axis $\mathbf{m}$ as in the harmonic homology. We say $\lambda$ is the factor of the centered collineation. For $\lambda = -1$, one obtains the harmonic homology; $\lambda = 1$ yields the identity; $\lambda = 0$, the projection onto $\mathbf{Z}$; $\lambda = \infty$, projection onto $\mathbf{m}$. Notice that $P_{Z,m,0}$ is not defined for $\mathbf{P} \in \mathbf{m}$, and $P_{Z,m,\infty}$ is not defined for $\mathbf{P} = \mathbf{Z}$.

**Theorem 12.** *For* $\mathbf{P}' = P_{Z,m,\lambda}(\mathbf{P})$, $(\mathbf{S}, \mathbf{Z}; \mathbf{P}, \mathbf{P}') = \lambda$.

*Proof.* WLOG we can assume $\mathbf{Z}$ is chosen so that $\langle \mathbf{Z}, \mathbf{m}\rangle = 1$. Define $x := \langle \mathbf{P}, \mathbf{m}\rangle$. Setting $\mathbf{S} = \alpha\mathbf{P} + \beta\mathbf{Z}$ and solving $\langle \mathbf{S}, \mathbf{m}\rangle = 0$ leads to $\mathbf{S} = \mathbf{P} - x\mathbf{Z}$. Assigning homogeneous coordinates $(0, 1)$ to $\mathbf{S}$ and $(1, 0)$ to $\mathbf{Z}$ leads then to coordinates $(x, 1)$ for $\mathbf{P}$ and $(x, \lambda)$ for $\mathbf{P}'$. Evaluating the cross ratio $(\mathbf{S}, \mathbf{Z}; \mathbf{P}, \mathbf{P}')$ using Def. 4 yields the desired result. □

*Remark 13.* There are various ways to parametrize the family $P_{Z,m,\lambda}$; the one given above is chosen since it behaves nicely with respect to orientation of fixed points. For example, for positive $\lambda$, the point-pairs $(\mathbf{P}, \mathbf{P}')$ and $(\mathbf{Z}, \mathbf{S})$ do not separate each other, so the collineation preserves order along each invariant line. Correspondingly, fixed points $\mathbf{P} \in \mathbf{m}$ are mapped to positive multiples of themselves; the opposite is true for negative $\lambda$. For $\lambda = 0$, $\mathbf{P}'$ is not defined, and for $\lambda = \pm\infty$, the freedom to choose the sign shows that it is impossible to define an orientation in this case also.

Similar remarks apply for $\mathbb{R}P^{2n}$; in odd dimensions there is no way to consistently assign orientation to the fixed points using the formula, since all fixed points are reversed by the centered collineation. A similar analysis shows that $\mathbf{Z}$ is always mapped to a positive multiple of itself; this reflects the fact that for all $\lambda$, a small neighborhood of $\mathbf{Z}$ is mapped to a small neighborhood of itself with the same orientation.

*Remark 14.* Like any collineation, the harmonic homology has an induced action on the dual space of hyperplanes. Viewed as a collineation of the dual space, this is also a harmonic homology, with center $\mathbf{m}$ and axis $\mathbf{Z}$. Thus, the concept of harmonic homology is a *self-dual* one. One obtains the dual formula by dualizing (2.1):

$$H_{Z,m}(\mathbf{l}) = -\langle \mathbf{Z}, \mathbf{m}\rangle \mathbf{l} + 2\langle \mathbf{Z}, \mathbf{l}\rangle \mathbf{m} \tag{2.3}$$

This dual version will be important when we take up this theme again in Sect. 4.6.

## 2.2 Exterior algebra

Let $\mathsf{V}$ be a real vector space of dimension $n$. The exterior, or Grassmann, algebra $\bigwedge(\mathsf{V})$, is generated by the exterior product[1] $\wedge$ applied to the vectors of $\mathsf{V}$. The exterior product is an alternating, bilinear operation. The algebra has a graded structure. The elements of grade-1 are defined to be the vectors of $\mathsf{V}$; the exterior product of a $k$- and $m$-vector is a $(k+m)$-vector, when the operands are linearly independent subspaces. An element that can be represented as a wedge product of $k$ 1-vectors is called a simple $k$-vector, or $k$-*blade*. The $k$-blades generate the vector subspace $\bigwedge^k(\mathsf{V})$, whose elements are said to have grade $k$. This subspace has dimension $\binom{n}{k}$, hence the total dimension of the exterior algebra is $2^n$.

**Simple and non-simple vectors.** A $k$-blade represents the subspace of $\mathsf{V}$ spanned by the $k$ vectors which define it. Hence, the exterior algebra contains within it a representation of the subspace lattice of $\mathsf{V}$. For $n > 3$ there are also $k$-vectors which are not blades and do not represent a subspace of $\mathsf{V}$. Such vectors occur as bivectors when $\mathsf{V} = \mathbb{R}^4$ and play an important role in the discussion of kinematics and dynamics in Chapter 8 and Chapter 9.

**Dual Grassmann algebra.** The same construction can be applied to construct $\bigwedge \mathsf{V}^*$, the exterior algebra of the dual vector space $\mathsf{V}^*$. This is the algebra of alternating $k$-multilinear forms.

### 2.2.1 Determinant function

$\bigwedge^n(\mathsf{V})$ is a one-dimensional vector space. Let $\mathbf{I}$ be a basis element. Given a basis $\{\mathbf{v}_i\}$ for $\mathsf{V}$, $\mathbf{v}_1 \wedge \mathbf{v}_2 ... \wedge \mathbf{v}_n \in \bigwedge^n(\mathsf{V})$, hence $\mathbf{v}_1 \wedge \mathbf{v}_2 ... \wedge \mathbf{v}_n = \alpha\mathbf{I}$ for some non-zero $\alpha \in \mathbb{R}$. Define a function

$$\triangle : \otimes^n \mathsf{V} \to \mathbb{R} \quad \text{by} \quad \triangle(\{\mathbf{v}_i\}) := \alpha$$

Then $\triangle$ is called the *determinant* function of $\bigwedge(\mathsf{V})$. It lets us define a canonical isomorphism between $\mathsf{V}$ and $\bigwedge^{n-1}(\mathsf{V}^*)$.

**Theorem 15.** $\mathsf{V} \cong \bigwedge^{n-1}(\mathsf{V}^*)$

*Proof.* Given $\mathbf{v} \in \mathsf{V}$, then define $\omega \in \bigwedge^{n-1}(\mathsf{V}^*)$ by

$$\omega(\mathbf{v}_1, \mathbf{v}_2, ...\mathbf{v}_{n-1}) := \triangle(\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_{n-1})$$

Conversely, given such an $\omega$, there is a unique $\mathbf{v}$ such that the above equation is satisfied. Hence $\mathsf{V} \cong \bigwedge^{n-1}(\mathsf{V}^*)$. □

*Remark 16.* By abstract nonsense, this implies $\mathsf{V}^* \cong \bigwedge^{n-1}(\mathsf{V})$.

[1] Also called the outer or wedge product

### 2.2.2 Projectivized exterior algebra

The exterior algebra can be projectivized using the same process defined above for the construction of $P(\mathsf{V})$ from $\mathsf{V}$, but applied to the vector spaces $\bigwedge^k(\mathsf{V})$. This yields the projectivized exterior algebra $W := \mathbf{P}(\bigwedge(\mathsf{V}))$. The operations of $\bigwedge(\mathsf{V})$ carry over to $\mathbf{P}(\bigwedge \mathsf{V})$, since, roughly speaking: "Projectivization commutes with outer product". That is, for two elements $X, Y \in \bigwedge(\mathsf{V})$:

$$\mathbf{P}(X) \wedge \mathbf{P}(Y) = \mathbf{P}(X \wedge Y)$$

The difference lies in how the elements and operations are projectively interpreted. The $k$-blades of $\mathbf{P}(\bigwedge \mathsf{V})$ correspond to $(k-1)$-dimensional subspaces of $\mathbf{P}(V)$. All multiples of the same $k$-blade represent the same projective subspace, and differ only by intensity ([Whi98], §16-17). 1-blades correspond to points; 2-blades to lines; 3-blades to planes, etc.

**2.2.2.1 Dual exterior algebra** The algebra $\mathbf{P}(\bigwedge \mathsf{V}^*)$ is formed by projectivizing the dual algebra $\bigwedge(\mathsf{V}^*)$. $\mathbf{P}(\bigwedge \mathsf{V}^*)$ is the alternating algebra of $k$-multilinear forms. By abstract nonsense, $\mathbf{P}(\bigwedge \mathsf{V}^*) = (\mathbf{P}(\bigwedge \mathsf{V}))^*$: projectivization commutes with dualization. $\mathbf{P}(\bigwedge \mathsf{V}^*)$ is naturally isomorphic to $\mathbf{P}(\bigwedge \mathsf{V})$; again, the difference lies in how the elements and operations are interpreted. Like $\mathbf{P}(\bigwedge \mathsf{V})$, $\mathbf{P}(\bigwedge \mathsf{V}^*)$ represents the subspace structure of $\mathbf{P}(\mathsf{V})$, but turned on its head: 1-vectors represent projective hyperplanes, while simple (n-1)-vectors represent projective points. The outer product $\mathbf{a} \wedge \mathbf{b}$ corresponds to the *meet* rather than *join* operator. See also Fig. 2.4.

**2.2.2.2 Notation alert** In order to distinguish the two outer products of $\mathbf{P}(\bigwedge \mathsf{V})$ and $\mathbf{P}(\bigwedge \mathsf{V}^*)$, we write the outer product in $\mathbf{P}(\bigwedge \mathsf{V})$ as $\vee$, and leave the outer product in $\mathbf{P}(\bigwedge \mathsf{V}^*)$ as $\wedge$. These symbols match closely the affiliated operations of join (union $\cup$) and meet (intersection $\cap$), resp. Note, however, they are reversed from some modern literature ([HZ91]).

### 2.2.3 Exterior power of a map

Given a linear map $f : \mathsf{V} \to \mathsf{V}$, there is an induced grade-preserving map $\bigwedge(f) : \bigwedge(\mathsf{V}) \to \bigwedge(\mathsf{V})$ called the *exterior power* of $f$. Its action on a simple $k$-vector $\mathbf{a} = \mathbf{e}_{i_1} \wedge ... \wedge \mathbf{e}_{i_k}$ is defined by

$$\wedge^k(f) = f(\mathbf{e}_{i_1}) \wedge ... \wedge f(\mathbf{e}_{i_k} \tag{2.4}$$

For $f : \mathsf{V} \to \mathsf{V}^*$, one defines a map $\bigwedge(f) : \bigwedge(\mathsf{V}) \to \bigwedge(\mathsf{V}^*)$ by using the wedge in the dual algebra in the RHS of (2.4).

*Remark 17.* $\wedge^n(f)$ gives the determinant of the matrix of $f$ when $f$ is expressed in terms of a basis $\{\mathbf{v}_i\}$ satisfying $\triangle(\{\mathbf{v}_i\}) = 1$.

**2.2.3.1 The adjoint map** Given $f : \mathsf{V} \to \mathsf{V}^*$, construct the exterior power $\bigwedge^{n-1}(f) : \bigwedge(\mathsf{V}) \to \bigwedge(\mathsf{V}^*)$. By Sect. 2.2.1, $\bigwedge^{n-1}(f)$ can be considered as a map $\mathsf{V}^* \to \mathsf{V}$. It is called the *adjoint* of $f$. We write $f^* := \bigwedge^{n-1}(f)$. With respect to a basis, the matrix of $f^*$ is the "matrix of cofactors" of the matrix of $f$, which isn't surprising considering the role played in its definition by the $\triangle$ function. For invertible $f$, $f^*$ is the unique linear map satisfying $\langle \mathbf{u}, \mathbf{x} \rangle = \langle f(\mathbf{x}), f^*(\mathbf{u}) \rangle$.

*Remark 18.* The adjoint is sometimes defined by identifying $\mathsf{V}$ and $\mathsf{V}^*$ using a metric. See for example [DFM07], Sec. 4.3.2. We prefer to avoid the use of metrics where they are not required. See related discussion in Sect. 5.10.

### 2.2.4 Equal rights for $\mathbf{P}(\bigwedge \mathsf{V})$ and $\mathbf{P}(\bigwedge \mathsf{V}^*)$

From the point of view of representing $\mathsf{V}$, $\mathbf{P}(\bigwedge \mathsf{V})$ and $\mathbf{P}(\bigwedge \mathsf{V}^*)$ are equivalent. There is no *a priori* reason to prefer one to the other. Every geometric element in one algebra occurs in the other, and any configuration in one algebra has a dual configuration in the other obtained by applying the Principle of Duality [Cox87], to the configuration. We refer to $\mathbf{P}(\bigwedge \mathsf{V})$ as a *point-based*, and $\mathbf{P}(\bigwedge \mathsf{V}^*)$ as a *plane-based*, algebra.[2]

Depending on the context, one or the other of the two algebras may be more useful. Here are some examples:

1. **Joins and meets.** $\mathbf{P}(\bigwedge \mathsf{V})$ is the natural choice to calculate subspace joins, and $\mathbf{P}(\bigwedge \mathsf{V}^*)$, to calculate subspace meets. See Sect. 2.3.1.4.
2. **Spears and axes.** Lines appear in two aspects: as spears (bivectors in $\mathbf{P}(\bigwedge \mathsf{V})$) and axes (bivectors in $\mathbf{P}(\bigwedge \mathsf{V}^*)$). See Sect. 2.2.4.1.
3. **Euclidean geometry.** $\mathbf{P}(\bigwedge \mathsf{V}^*)$ is the correct choice to use for modeling euclidean geometry. See Sect. 5.3.
4. **Reflections in planes.** $\mathbf{P}(\bigwedge \mathsf{V}^*)$ has advantages for kinematics, since it naturally allows building up rotations as products of reflections in planes. See Sect. 5.6.1.

We turn now to item 2 above, highlighting the importance of maintaining $\mathbf{P}(\bigwedge \mathsf{V})$ and $\mathbf{P}(\bigwedge \mathsf{V}^*)$ as equal citizens.

**2.2.4.1 There are no lines, only spears and axes!** Most of this work is focused on the case $\mathsf{V} = \mathbb{R}^4$. In this case, bivectors are self-dual. This has interesting consequences for how they are interpreted.

Given two points $\mathbf{x}$ and $\mathbf{y} \in \mathbf{P}(\bigwedge \mathsf{V})$, the condition that a third point $\mathbf{z}$ lies in the subspace spanned by the 2-blade $\mathbf{l} := \mathbf{x} \vee \mathbf{y}$ is that $\mathbf{x} \vee \mathbf{y} \vee \mathbf{z} = 0$, which implies that $\mathbf{z} = \alpha\mathbf{x} + \beta\mathbf{y}$ for some $\alpha, \beta$ not both zero. In projective geometry, such a set is called a *point range*. We prefer the more colorful term *spear*. Dually, given two planes $\mathbf{x}$ and $\mathbf{y} \in W^*$, the condition that a third

[2] We prefer the dimension-dependent formulation *plane-based* to the more precise *hyperplane-based*. We also prefer not to refer to the plane-based algebra as the *dual* algebra, since this formulation depends on the accident that the original algebra is interpreted as point-based.

plane $\mathbf{z}$ passes through the subspace spanned by the 2-blade $\mathbf{l} := \mathbf{x} \wedge \mathbf{y}$ is that $\mathbf{z} = \alpha\mathbf{x} + \beta\mathbf{y}$. In projective geometry, such a set is called a *plane pencil.* We prefer the more colorful term *axis*.

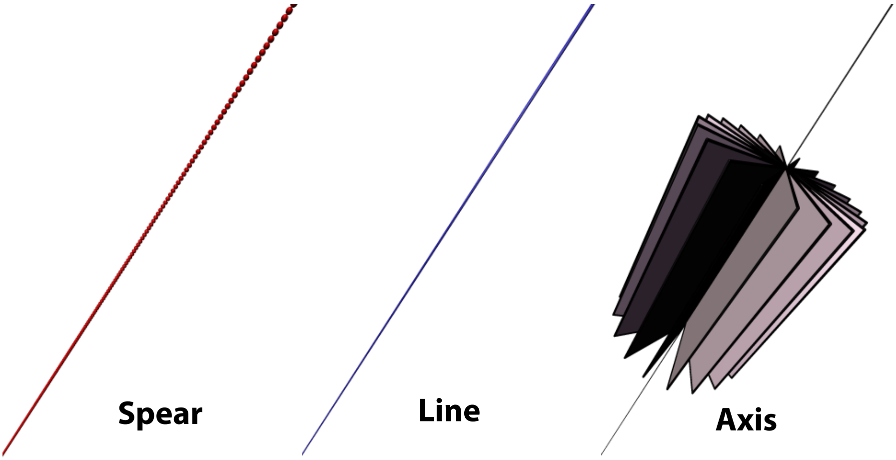


**Fig. 2.2** Three aspects of line: spear (all incident points); line *qua* line; and axis (all incident planes).

Within the context of $\mathbf{P}(\bigwedge \mathsf{V})$ and $\mathbf{P}(\bigwedge \mathsf{V}^*)$, lines exist only in one of these two aspects: of spear – as bivector in $\mathbf{P}(\bigwedge \mathsf{V})$ – and axis – as bivector in $\mathbf{P}(\bigwedge \mathsf{V}^*)$. This naturally generalizes to non-simple bivectors: there are point-wise bivectors (in $\mathbf{P}(\bigwedge \mathsf{V})$), and plane-wise bivectors (in $\mathbf{P}(\bigwedge \mathsf{V}^*)$.) Many of the important operators of geometry and dynamics we will meet below, such as the polarity on the metric quadric (Sect. 4.4), and the inertia tensor of a rigid body (Sect. 9.3), map $\langle \mathbf{P}(\bigwedge \mathsf{V}) \rangle_2$ to $\langle \mathbf{P}(\bigwedge \mathsf{V}^*) \rangle_2$ and hence map spears to axes and vice-versa. Having both algebras on hand preserves the qualitative difference between these dual aspects of the generic term "line".

*Remark 19.* It is possible to build up projective geometry by beginning with the line as the primitive element and constructing points and planes from this primitive element. This would then provide a third way to view a line, so to speak, in its own right rather than built out of points or planes. This approach for example can be found in [Sto09]. But this approach does not lend itself to representing the subspace structure of $\mathbb{R}P^3$ with Grassmann algebras.

## 2.3 Poincaré Duality

Our treatment differs from other approaches (for example, Grassmann-Cayley algebras) in explicitly maintaining both algebras on an equal footing rather than expressing the wedge product in one in terms of the wedge product of the other (as in the Grassman-Cayley *shuffle* product) ([Sel05], [Per09]). To switch back and forth between the two algebras, we construct an algebra isomorphism that, given an element of one algebra, produces the element of the second algebra which corresponds to the same geometric entity of $\mathsf{V}^*$.

This algebra isomorphism can be stated and proved in a coordinate-free way using advanced techniques of modern multilinear algebra ([Gre67b], Ch. 6, §2). In this form the isomorphism is called the *Poincaré isomorphism*, and the resulting equivalence, *Poincaré duality*. We derive it here using a particular coordinate system which simplifies the exposition. We first show how this works for the case of interest $\mathsf{V} = \mathbb{R}^4$.

### 2.3.1 The isomorphism J

Each weighted subspace $S$ of $\mathbb{R}P^3$ corresponds to a unique element $S_W$ of $\mathbf{P}(\bigwedge \mathsf{V})$ and to a unique element $S_{W^*}$ of $\mathbf{P}(\bigwedge \mathsf{V}^*)$. We seek a bijection $\mathbf{J}$ : $\mathbf{P}(\bigwedge \mathsf{V}) \leftrightarrow \mathbf{P}(\bigwedge \mathsf{V}^*)$ such that $J(S_W) = S_{W^*}$. If we have found $\mathbf{J}$ for the basis $k$-blades, then it extends by linearity to multivectors. This will be the desired Poincaré isomorphism. To that end, we introduce a basis for $\mathbb{R}^4$ and extend it to a basis for $\mathbf{P}(\bigwedge \mathsf{V})$ and $\mathbf{P}(\bigwedge \mathsf{V}^*)$ so that $\mathbf{J}$ takes a particularly simple form. Refer to Fig. 2.3.

**2.3.1.1 The canonical basis** A basis $\{\mathbf{e}^0, \mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3\}$ of $\mathbb{R}^4$ corresponds to a coordinate tetrahedron for $\mathbb{R}P^3$, with corners occupied by the basis elements[3]. Use the same names to identify the elements of $P(\bigwedge^1(\mathbb{R}^4))$ which correspond to these projective points. Further, let $\mathbf{I}^0 := \mathbf{e}^0 \wedge \mathbf{e}^1 \wedge \mathbf{e}^2 \wedge \mathbf{e}^3$ be the basis element of $P(\bigwedge^4(\mathbb{R}^4))$, and $\mathbf{1}^0$ be the basis element for $P(\bigwedge^0(\mathbb{R}^4))$. Let the basis for $P(\bigwedge^2(\mathbb{R}^4))$ be given by the six edges of the tetrahedron:

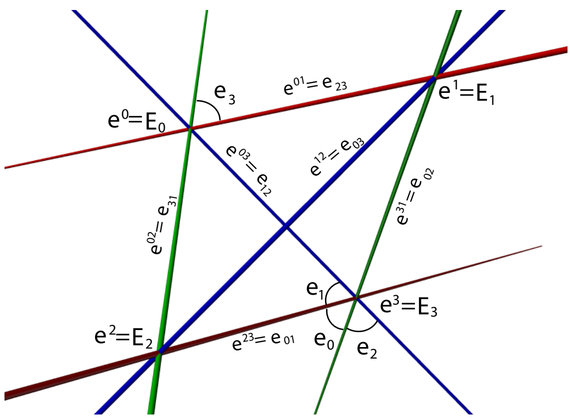


**Fig. 2.3** Fundamental tetrahedron with dual labeling. Entities in $W$ have superscripts; entities in $W^*$ have subscripts. Planes are identified by labeled angles of two spanning lines. A representative sampling of equivalent elements is shown.

[3] We use superscripts for $\mathbf{P}(\bigwedge \mathsf{V})$ and subscripts for $\mathbf{P}(\bigwedge \mathsf{V}^*)$ since $\mathbf{P}(\bigwedge \mathsf{V}^*)$ will be the more important algebra for our purposes.

$$\{\mathbf{e}^{01}, \mathbf{e}^{02}, \mathbf{e}^{03}, \mathbf{e}^{12}, \mathbf{e}^{31}, \mathbf{e}^{23}\}$$

where $\mathbf{e}^{ij} := \mathbf{e}^i \wedge \mathbf{e}^j$ represents the oriented line joining $\mathbf{e}^i$ and $\mathbf{e}^j$.[4] Finally, choose a basis $\{\mathbf{E}^0, \mathbf{E}^1, \mathbf{E}^2, \mathbf{E}^3\}$ for $P(\bigwedge^3(\mathbb{R}^4))$ satisfying the condition that $\mathbf{e}^i \vee \mathbf{E}^i = \mathbf{I}^0$. This corresponds to choosing the $i^{th}$ basis 3-vector to be the plane opposite the $i^{th}$ basis 1-vector in the fundamental tetrahedron, oriented in a consistent way.

| feature | $\mathbf{P}(\bigwedge \mathsf{V})$ | $\mathbf{P}(\bigwedge \mathsf{V}^*)$ |
|---|---|---|
| 0-vector | scalar $\mathbf{1}^0$ | scalar $\mathbf{1}_0$ |
| vector | point $\{e^i\}$ | plane $\{e_i\}$ |
| bivector | "spear" $\{e^{ij}\}$ | "axis" $\{e_{ij}\}$ |
| trivector | plane $\{E^i\}$ | point $\{E_i\}$ |
| 4-vector | $\mathbf{I}^0$ | $\mathbf{I}_0$ |
| outer product | join $\vee$ | meet $\wedge$ |

**Table 2.1** Comparison of $\mathbf{P}(\bigwedge \mathsf{V})$ and $\mathbf{P}(\bigwedge \mathsf{V}^*)$ for $\mathsf{V} = \mathbb{R}^4$.

We repeat the process for the algebra $\mathbf{P}(\bigwedge \mathsf{V}^*)$, writing indices as subscripts. Choose the basis 1-vector $\mathbf{e}_i$ of $\mathbf{P}(\bigwedge \mathsf{V}^*)$ to represent the same plane as $\mathbf{E}^i$. That is, $\mathbf{J}(\mathbf{E}^i) = \mathbf{e}_i$. Let $\mathbf{I}_0 := \mathbf{e}_0 \wedge \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$ be the pseudoscalar of the algebra. Construct bases for grade-0, grade-2, and grade-3 using the same rules as above for $\mathbf{P}(\bigwedge \mathsf{V})$ (i. e., replacing subscripts by superscripts). The results are represented in Table 2.1.

Given this choice of bases for $\mathbf{P}(\bigwedge \mathsf{V})$ and $\mathbf{P}(\bigwedge \mathsf{V}^*)$, examination of Fig. 2.3 makes clear that, on the basis elements, $\mathbf{J}$ takes the following simple form:

$$\mathbf{J}(e^i) := E_i, \quad \mathbf{J}(E^i) := e_i, \quad \mathbf{J}(e^{ij}) := e_{kl} \tag{2.5}$$

where in the last equation, $(ijkl)$ an even permutation of $(0123)$.

Fig. 2.4 gives a graphical representation of Table 2.1, and the isomorphism $\mathbf{J}$.

**2.3.1.2 Description of J** Furthermore, $\mathbf{J}(\mathbf{1}^0) = \mathbf{I}_0$ and $\mathbf{J}(\mathbf{I}^0) = \mathbf{1}_0$ since these grades are one-dimensional. To sum up: the map $\mathbf{J}$ is grade-reversing and, considered as a map of coordinate-tuples, it is the identity map on all grades except for bivectors. What happens for bivectors? In $\mathbf{P}(\bigwedge \mathsf{V})$, consider $\mathbf{e}^{01}$, the joining line of points $\mathbf{e}^0$ and $\mathbf{e}^1$ (refer to Fig. 2.3). In $\mathbf{P}(\bigwedge \mathsf{V}^*)$, the same line is $\mathbf{e}_{23}$, the intersection of the only two planes which contain both of these points, $\mathbf{e}_2$ and $\mathbf{e}_3$. On a general bivector, $\mathbf{J}$ takes the form:

$$\mathbf{J}(a_{01}e^{01} + a_{02}e^{02} + a_{03}e^{03} + a_{12}e^{12} + a_{31}e^{31} + a_{23}e^{23}) =$$
$$a_{23}e_{01} + a_{31}e_{02} + a_{12}e_{03} + a_{03}e_{12} + a_{02}e_{31} + a_{01}e_{23}$$

[4] Note that the orientation of $\mathbf{e}^{31}$ is reversed; this is traditional since Plücker introduced these line coordinates.

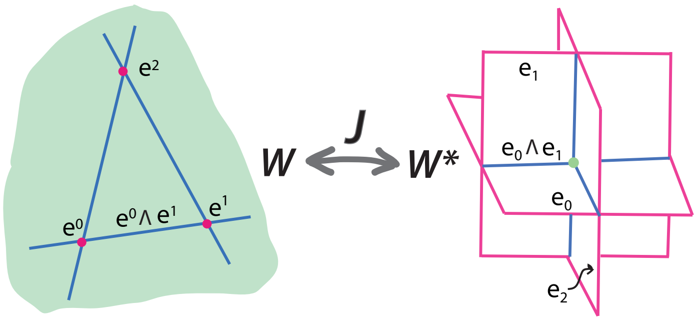


**Fig. 2.4** The standard Grassmann $\mathbf{P}(\bigwedge \mathsf{V})$ and its dual $\mathbf{P}(\bigwedge \mathsf{V}^*)$ are related by the Poincare isomorphism $\mathbf{J}$.

The coordinate-tuple is reversed. See Fig. 2.2. This behavior is characteristic of the situation in higher dimension, to which we now turn.

**2.3.1.3 J in $n$-dimensions** Here we generalize the construction above for $n = 4$ to arbitrary dimension, to show how to construct the algebra isomorphism $\mathbf{J}$ with the desired property, and connect it to the principle of *Poincaré duality*. We take up the issue of $\mathbf{J}$ again in Sect. 5.10 where we discuss it in relation to alternative formulations involving a metric.

A subset $S = \{i_1, i_2, ...i_k\}$ of $N = \{1, 2, ..., n\}$ is called a *canonical* k-*tuple* of $N$ if $i_1 < i_2 < ... < i_k$. For each canonical $k$-tuple of $N$, define $S^\perp$ to be the canonical $(n-k)$-tuple consisting of the elements $N \setminus S$. For each unique pair $\{S, S^\perp\}$, swap a pair of elements of $S^\perp$ if necessary so that the concatenation $SS^\perp$, as a permutation $P$ of $N$, is even. Call the collection of the resulting sets $\mathfrak{S}$. For each $S \in \mathfrak{S}$, define $\mathbf{e}^S = \mathbf{e}^{i_1}...\mathbf{e}^{i_k}$. We call the resulting set $\{\mathbf{e}^S\}$ the *canonical basis* for $\mathbf{P}(\bigwedge \mathsf{V})$ generated by $\{\mathbf{e}^i\}$.

Consider $\mathbf{P}(\bigwedge \mathsf{V}^*)$, the dual algebra to $\mathbf{P}(\bigwedge \mathsf{V})$. Choose a basis $\{\mathbf{e}_1, \mathbf{e}_2, ...\mathbf{e}_n\}$ for $\mathbf{P}(\bigwedge^1 \mathsf{V}^*)$ so that $\mathbf{e}_i$ represents the same oriented subspace as the basis *(n-1)*-vector $\mathbf{e}^{(i^\perp)}$ of $\mathbf{P}(\bigwedge \mathsf{V})$ represents. Construct the canonical basis (as above) of $\mathbf{P}(\bigwedge \mathsf{V}^*)$ generated by the basis $\{\mathbf{e}_i\}$. Then define a map $\mathbf{J} : \mathbf{P}(\bigwedge \mathsf{V}) \to \mathbf{P}(\bigwedge \mathsf{V}^*)$ by $\mathbf{J}(\mathbf{e}^S) = \mathbf{e}_{S^\perp}$ and extend by linearity.

$\mathbf{J}$ is an "identity" map on the subspace structure of $\mathsf{V}$: it maps a simple $k$-vector $B \in W$ to the simple $(n-k)$-vector $\in \mathbf{P}(\bigwedge \mathsf{V}^*)$ which represents the same geometric entity as $B$ does in $\mathbb{R}P^n$. *Proof:* By construction, $\mathbf{e}^S$ represents the join of the 1-vectors $\mathbf{e}^{i_j}, (i_j \in S)$ in $W$. This is however the same subspace as the meet of the $n-k$ basis 1-vectors $\mathbf{e}_{i_j}, (i_j \in S^\perp)$ of $\mathbf{P}(\bigwedge \mathsf{V}^*)$, since $\mathbf{e}_i$ is incident with $\mathbf{e}^j \iff j \neq i$.

From the construction of $\mathbf{J}$ we can consider it as a grade-reversing isomorphism $\mathbf{P}(\bigwedge \mathsf{V}) \leftrightarrow \mathbf{P}(\bigwedge \mathsf{V}^*)$ such that $\mathbf{J}^2 = id$. Strictly speaking, this is the identity only projectively, in a strict vector space interpretation, $\mathbf{J}(\mathbf{J}(X)) = \pm\mathbf{X}$ for an arbitrary $k-$blade $\mathbf{X}$.

The full significance of $\mathbf{J}$ will only become evident after metrics are introduced. See Sect. 5.10.

We now show how to use $\mathbf{J}$ to define meet and join operators valid for both $\mathbf{P}(\bigwedge \mathsf{V})$ and $\mathbf{P}(\bigwedge \mathsf{V}^*)$.

**2.3.1.4 Projective join and meet** Knowledge of $\mathbf{J}$ allows equal access to join and meet operations. We define a meet operation $\wedge$ for two blades $A, B \in \mathbf{P}(\bigwedge \mathsf{V})$:

$$A \wedge B = \mathbf{J}(\mathbf{J}(A) \wedge \mathbf{J}(B)) \tag{2.6}$$

and extend by linearity to the whole algebra. There is a similar expression for the join $\vee$ operation for two blades $A, B \in \mathbf{P}(\bigwedge \mathsf{V}^*)$:

$$A \vee B := \mathbf{J}(\mathbf{J}(A) \vee \mathbf{J}(B)) \tag{2.7}$$

## 2.4 Remarks on homogeneous coordinates

We use the terms *homogeneous* model and *projective* model interchangeably, to denote the projectivized version of Grassmann (and, later, Clifford) algebra.

The projective model allows a certain freedom in specifying results within the algebra. In particular, when the calculated quantity is a subspace, then the answer is only defined up to a non-zero scalar multiple. In some literature, this fact is represented by always surrounding an expression $x$ in square brackets $[x]$ when one means "the projective element corresponding to the vector space element $x$". Similarly, $\mathbf{x}\mathbb{R}$ is used to represent "the 1-dimensional vector subspace corresponding to the projective point $\mathbf{x}$". We do not adhere to this level of rigor here, since in most cases the intention is clear.

Some of the formulas introduced below take on a simpler form which take advantage of this freedom, but they may appear unfamiliar to those used to working in the more strict vector-space environment. On the other hand, when the discussion later turns to kinematics and dynamics, then this projective equivalence is no longer strictly valid. Different representatives of the same subspace represent weaker or stronger instances of a velocity or momentum (to mention two possibilities). In such situations terms such as *weighted* point or "point with intensity" will be used. See [Whi98], Book III, Ch. 4. See also Sect. 9.2.3.1 below, which discusses the use of homogeneous coordinates with respect to the inertia tensor of a rigid body.

## 2.5 Guide to the literature

[PW01] (Chapter 1 and Section 2.2) provides a good overview of the background material on projective geometry and exterior algebra. For detailed background on exterior algebras, see [Wik], [Bou89], or [Gre67b]. For more on Poincaré duality, consult [Gre67b]., Sec. 6.8. [Kow09] provides a good introduction to projective geometry with a synthetic component.